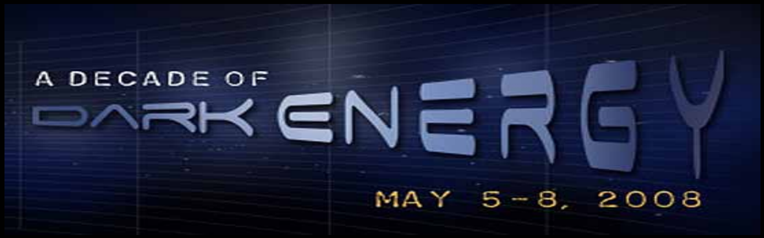
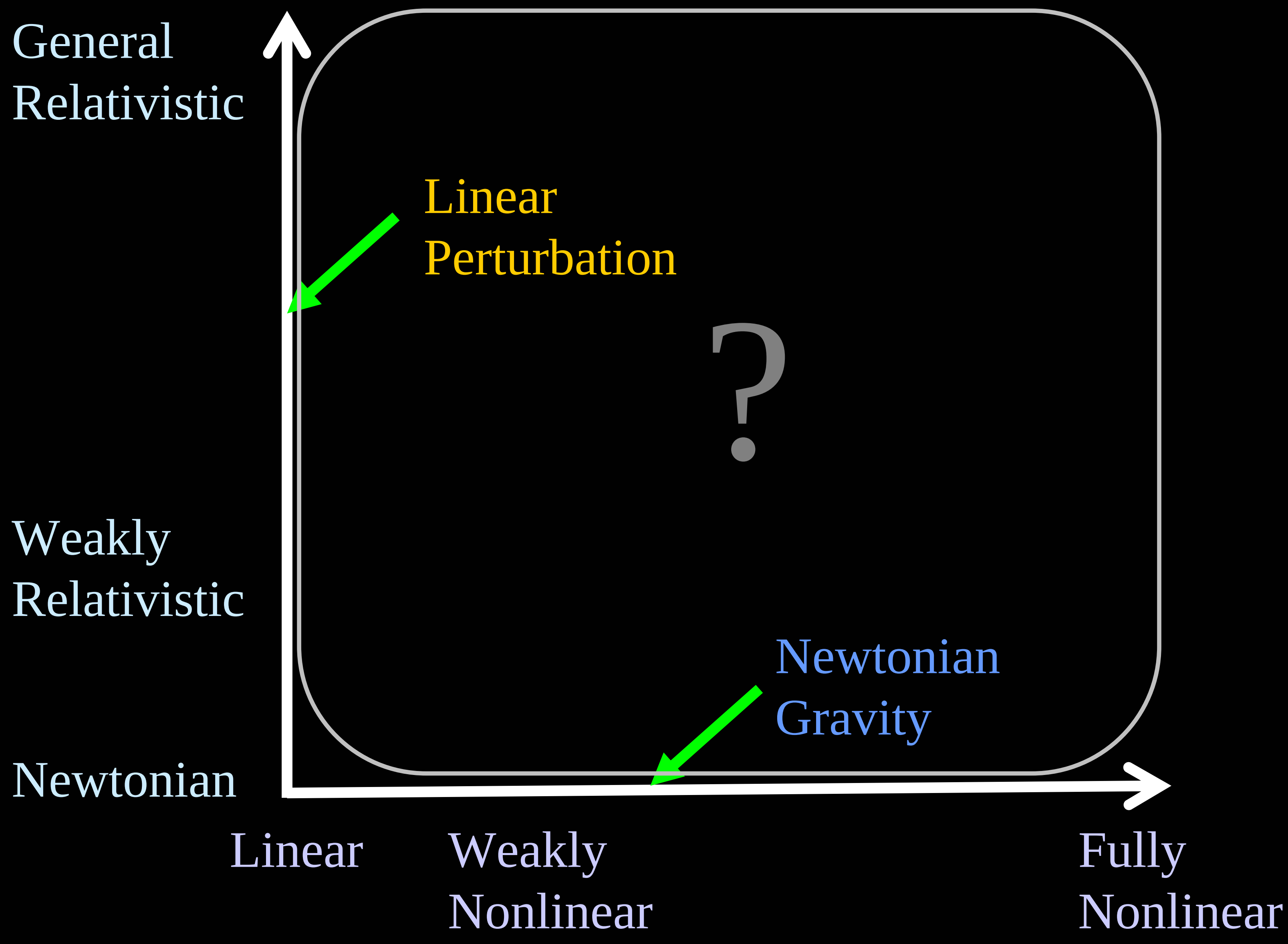
Cosmological Post-Newtonian Approximation with Dark Energy

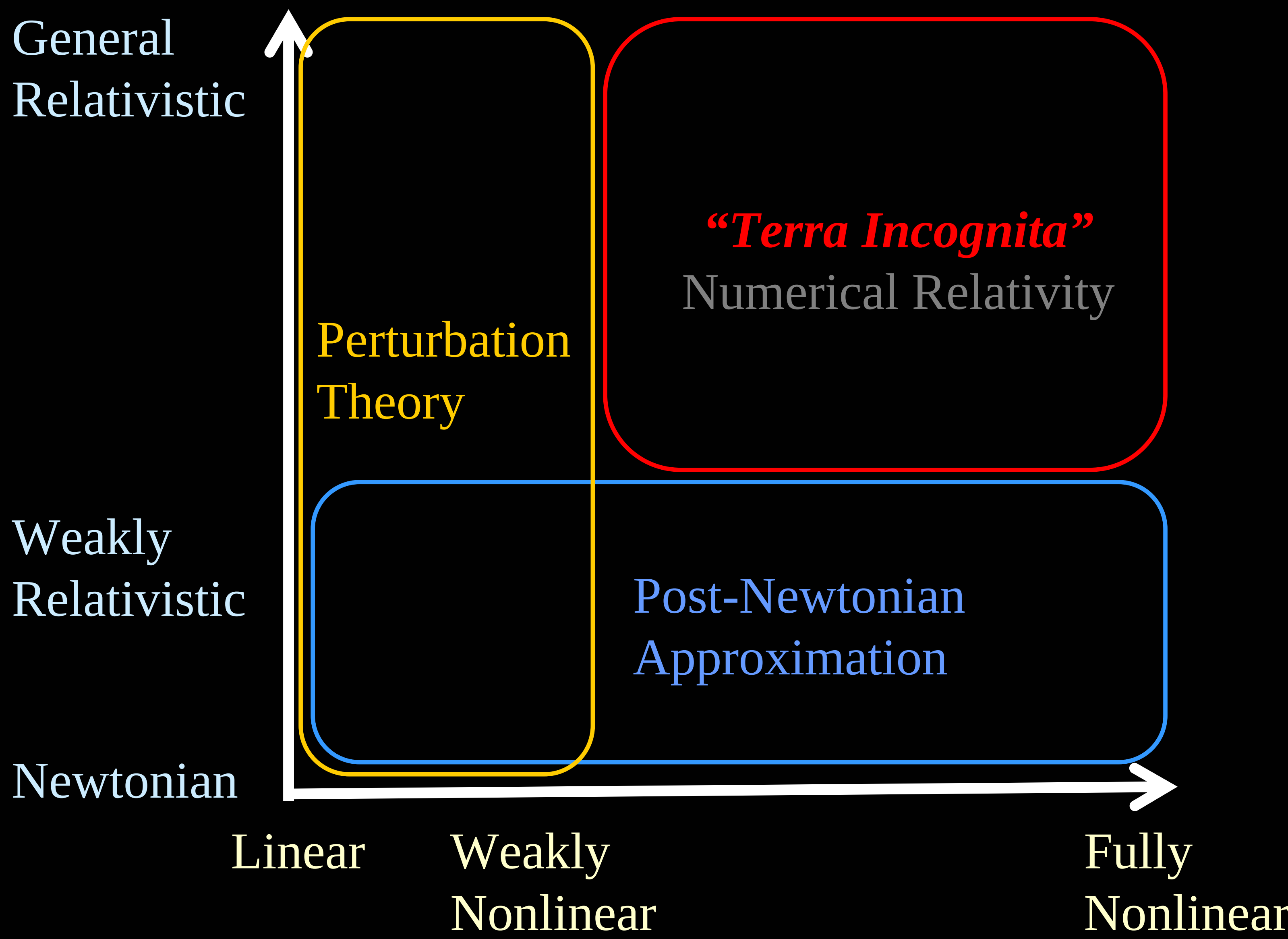


J. Hwang and H. Noh
2008.05.05

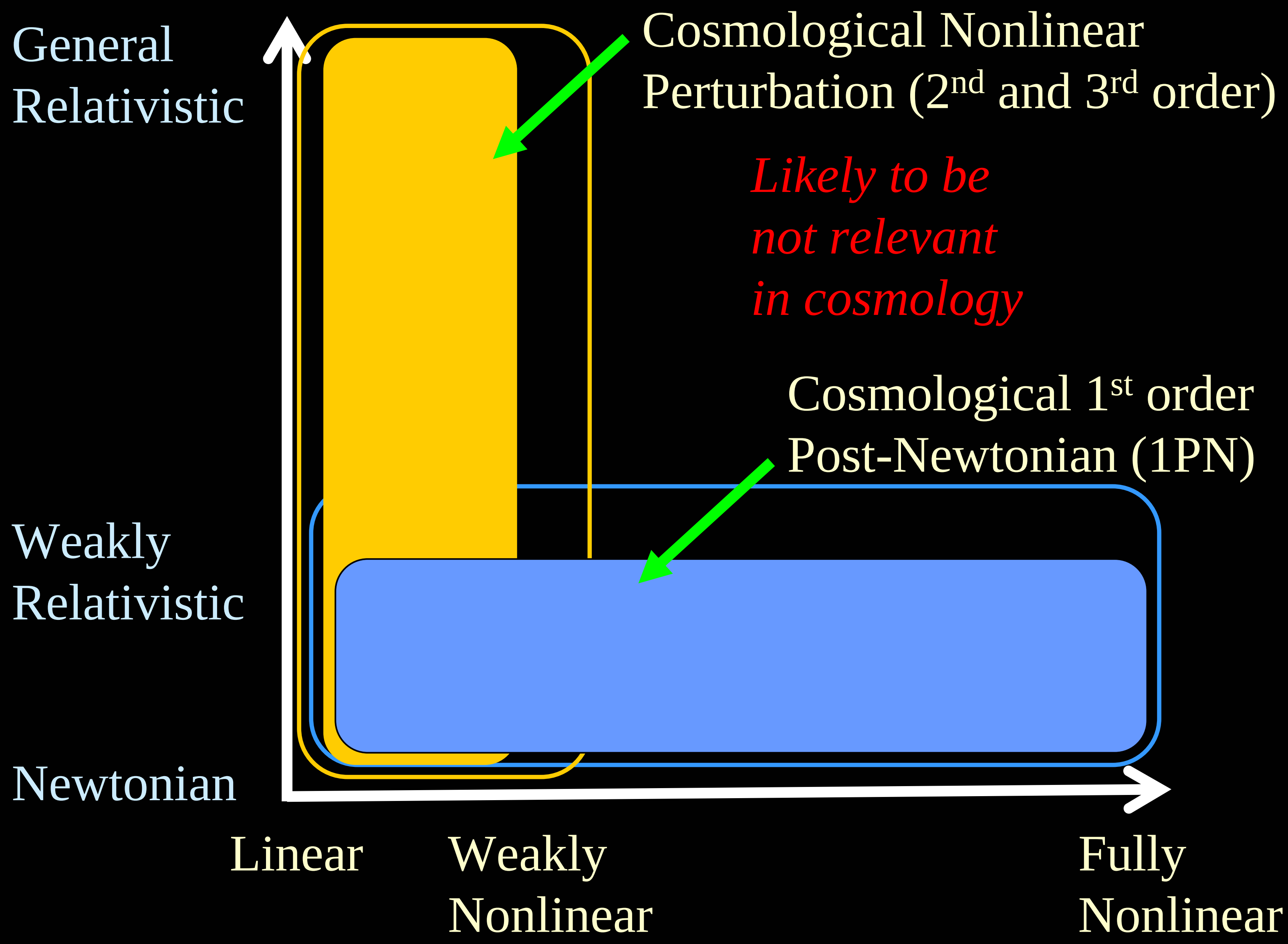
Studies of Large-scale Structure



Perturbation Theory *vs.* Post-Newtonian



Cosmology and Large-Scale Structure



Perturbation method:

- Perturbation expansion.
- All perturbation variables are small.
- Weakly nonlinear.
- Strong gravity; fully relativistic!
- Valid in all scales!

Post-Newtonian method:

- Abandon geometric spirit of GR: recover the good old absolute space and absolute time.
- Provide GR correction terms in the Newtonian equations of motion.
- Expansion in $\frac{GM}{Rc^2} \sim \frac{v^2}{c^2} << 1$
- Fully nonlinear!
- No strong gravity situation; weakly relativistic.
- Valid far inside horizon

Λ

- Cosmological constant, Λ, included.
- Λ appears only in the background equations!

Relativistic/Newtonian correspondence:

Background order:

Spatial curvature / Total energy

Cosmological constant

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\varrho - \frac{\mathrm{const}}{a^2} + \frac{\Lambda c^2}{3},$$

Friedmann (1922)/Milne and McCrea (1934)

Linear perturbation:

Density

Density perturbation

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G \varrho \delta = 0,$$

Lifshitz (1946)/Bonnor (1957)

PHYSICAL REVIEW D **69**, 104011 (2004)

Second-order perturbations of the Friedmann world model

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Newtonian equations: Peebles (1980) Fully nonlinear!

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\bar{\varrho}\delta = -\frac{1}{a^2}[a\nabla\cdot(\delta\mathbf{u})]^{\cdot} + \frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}). \tag{343}$$

Relativistic equations: Noh-Hwang (2004) To second order! K=0

$$\ddot{\delta}_v + 2H\dot{\delta}_v - 4\pi G\bar{\mu}\delta_v = -\frac{1}{a^2}[a\nabla\cdot(\delta_v\mathbf{u})]^{\cdot} + \frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) + \dot{C}^{(t)}_{\alpha\beta}\left(\frac{2}{a}\nabla^\alpha u^\beta + \dot{C}^{(t)\alpha\beta}\right). \tag{342}$$

comoving gauge

Gravitational waves

PHYSICAL REVIEW D **72,** 044012 (2005)

Third-order perturbations of a zero-pressure cosmological medium: Pure general relativistic nonlinear effects

Jai-chan Hwang[1] and Hyerim Noh[2]

Physical Review D, **72**, 044012 (2005).

Linear order:

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta = 0,$$

Second order: K = 0

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta = -\frac{1}{a^2}\frac{\partial}{\partial t}\left[a\nabla\cdot(\delta\mathbf{u})\right] + \frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}),$$

Third order: K = 0

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta = -\frac{1}{a^2}\frac{\partial}{\partial t}\left[a\nabla\cdot(\delta\mathbf{u})\right] + \frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u})$$
$$+ \frac{1}{a^2}\frac{\partial}{\partial t}\left\{a\left[2\varphi\mathbf{u} - \nabla(\Delta^{-1}X)\right]\cdot\nabla\delta\right\} - \frac{4}{a^2}\nabla\cdot\left[\varphi\left(\mathbf{u}\cdot\nabla\mathbf{u} - \frac{1}{3}\mathbf{u}\nabla\cdot\mathbf{u}\right)\right]$$
$$+ \frac{2}{3a^2}\varphi\mathbf{u}\cdot\nabla(\nabla\cdot\mathbf{u}) + \frac{\Delta}{a^2}\left[\mathbf{u}\cdot\nabla(\Delta^{-1}X)\right] - \frac{1}{a^2}\mathbf{u}\cdot\nabla X - \frac{2}{3a^2}X\nabla\cdot\mathbf{u},$$

$$X \equiv 2\varphi\nabla\cdot\mathbf{u} - \mathbf{u}\cdot\nabla\varphi + \frac{3}{2}\Delta^{-1}\nabla\cdot\left[\mathbf{u}\cdot\nabla(\nabla\varphi) + \mathbf{u}\Delta\varphi\right].$$

Pure General Relativistic corrections

Physical Review D, **72**, 044012 (2005).

φ

$$ds^2 = -a^2(1+2\alpha)d\eta^2 - 2a^2\beta_{,\alpha}d\eta dx^\alpha + a^2[g^{(3)}_{\alpha\beta}(1+2\varphi) + 2\gamma_{,\alpha|\beta} + 2C^{(t)}_{\alpha\beta}]dx^\alpha dx^\beta,$$

To linear order:

Curvature perturbation

$$R^{(h)} = \frac{6\bar{K}}{a^2} - 4\frac{\Delta + 3\bar{K}}{a^2}\varphi,$$

In the comoving gauge, flat background (including Λ):

$$\dot{\varphi}_v = 0. \qquad \varphi_v = C,$$

Curvature perturbation in the comoving gauge

CMB:

Curvature perturbation in the zero-shear gauge

$$\frac{\delta T}{T} \sim \frac{1}{3}\varphi_\chi = \frac{1}{3}\frac{\delta\Phi}{c^2} \sim \frac{1}{5}\varphi_v \sim \frac{1}{5}C \sim 10^{-5}$$

Sachs-Wolfe effect

COBE, WMAP

$$\varphi_v \sim 5 \times 10^{-5},$$

Physical Review D, **72**, 044012 (2005)

Mon. Not. R. Astron. Soc. **367,** 1515–1520 (2006) doi:10.1111/j.1365-2966.2006.10067.x

Why Newtonian gravity is reliable in large-scale cosmological simulations

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1. Relativistic/Newtonian correspondence to the second order
2. Pure general relativistic third-order corrections are small ~$5\text{x}10^{-5}$
3. Correction terms are independent of presence of the horizon.

Density power spectrum to second-order:

K=0 =Λ:

$$|\delta(\mathbf{k},t)|^2 = |\delta_1(\mathbf{k},t)|^2 + \frac{1}{(2\pi)^3}\int d^3k' \Big\{ \frac{2}{14^2}|\delta_1(\mathbf{k}',t)|^2|\delta_1(\mathbf{k}-\mathbf{k}',t)|^2 J^2(\mathbf{k},\mathbf{k}',\mathbf{k}-\mathbf{k}')$$
$$+|\delta_1(\mathbf{k},t)|^2\Big[|\delta_1(\mathbf{k}',t)|^2\Big(\frac{2}{63}F(\mathbf{k},\mathbf{k}',\mathbf{k}-\mathbf{k}')L(\mathbf{k}-\mathbf{k}',\mathbf{k},-\mathbf{k}')$$
$$+\frac{1}{18}H(\mathbf{k},\mathbf{k}')J(\mathbf{k}-\mathbf{k}',\mathbf{k},-\mathbf{k}') + \frac{1}{18}H(\mathbf{k},\mathbf{k}-\mathbf{k}')L(\mathbf{k}-\mathbf{k}',\mathbf{k},-\mathbf{k}')\Big) + C^+\Big]\Big\}$$

Newtonian
($\delta_2 \cdot \delta_2$, $\delta_1 \cdot \delta_3$)

$$+\frac{10}{21}\left(\frac{\ell}{\ell_H}\right)^2 |\delta_1(\mathbf{k},t)|^2 \frac{1}{(2\pi)^3}\int d^3k' \Big\{ |\delta_1(\mathbf{k}',t)|^2 \left(\frac{13}{3} + 4\frac{k^2}{k'^2} + 7\frac{\mathbf{k}\cdot\mathbf{k}'}{k'^2} - 12\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k'^4} + 14\frac{k^2}{k'^2}\frac{\mathbf{k}\cdot\mathbf{k}'}{k'^2}\right)$$
$$+\Big[|\delta_1(\mathbf{k}',t)|^2 M(\mathbf{k}-\mathbf{k}',\mathbf{k},-\mathbf{k}')\frac{k^2}{|\mathbf{k}-\mathbf{k}'|^2}\left(1 - 3\frac{\mathbf{k}\cdot\mathbf{k}'}{k'^2} + \frac{15}{2}\frac{\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|^2} + 3\frac{k^2}{k'^2}\frac{\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|^2}\right) + C^+\Big]\Big\},$$

Pure General Relativistic corrections ($\delta_1 \cdot \delta_3$)

$$M(\mathbf{k},\mathbf{k}',\mathbf{k}-\mathbf{k}') \equiv \frac{k^2}{k'^2} + \frac{k^2}{|\mathbf{k}-\mathbf{k}'|^2} + \frac{3}{4}\frac{\mathbf{k}\cdot\mathbf{k}'}{k'^2} + \frac{3}{4}\frac{\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|^2} - \frac{1}{4}\frac{k^2}{k'^2}\frac{\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|^2}.$$

$$\ell/\ell_H \equiv \dot{a}/(kc)$$

Vishniac (1983)/Noh-Hwang (2008)

Physical Review D, Submitted (2008)

- Background world model:

  Relativistic: Friedmann (1922)

  Newtonian: Milne-McCrea (1934)

  Coincide for zero-pressure

- Linear structures:

  Relativistic: Lifshitz (1946)

  Newtonian: Bonnor (1957)

  Coincide for zero-pressure

- Second-order structures:

  Newtonian: Peebles (1980)

  Relativistic: Noh-Hwang (2004)

  Coincide for zero-pressure, no-rotation

- Third-order structures: Relativistic: Hwang-Noh (2005)

  Pure general relativistic corrections

  $\delta T/T \sim 10^{-5}$ factor smaller, independent of horizon

Assumptions:

Our relativistic/Newtonian correspondence includes Λ, but assumes:

1. Flat Friedmann background
2. Zero-pressure
3. Irrotational
4. Single component fluid
5. No gravitational waves
6. Second order in perturbations

Relaxing any of these assumptions could lead to pure general relativistic effects!

PHYSICAL REVIEW D **76,** 103527 (2007)

Second-order perturbations of cosmological fluids: Relativistic effects of pressure, multicomponent, curvature, and rotation

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Einstein's gravity corrections to Newtonian cosmology:

1. Relativistic/Newtonian correspondence for a zero-pressure, irrotational fluid in flat background without gravitational waves.

2. Gravitational waves → Corrections

3. Third-order perturbations → Corrections
   → Small, independent of horizon

4. Background curvature → Corrections

5. Pressure → Relativistic even to the background and linear order

6. Rotation → Corrections
   → Newtonian correspondence in the small-scale limit

7. Multi-component zero-pressure irrotational fluids
   → Newtonian correspondence

8. Multi-component, third-order perturbations → Corrections
   → Small, independent of horizon

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Cosmological non-linear hydrodynamics with post-Newtonian corrections

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JCAP03

Minkowski background

Cosmological post-Newtonian metric (Chandrasekhar 1965):

$$ds^2 = -\left[1 - \frac{1}{c^2}2U + \frac{1}{c^4}\left(2U^2 - 4\Phi\right)\right] c^2\,dt^2 - \frac{1}{c^2}2aP_i\,dt\,dx^i + a^2\left(1 + \frac{1}{c^2}2V\right)\gamma_{ij}\,dx^i\,dx^j.$$

Robertson-Walker background metric

JCAP03 (2008) 010

Newtonian limit

Newtonian metric:

$$\mathrm{d}s^2 = -\left(1 - \frac{1}{c^2}2U\right)c^2\,\mathrm{d}t^2 + a^2\gamma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j.$$

- Newtonian gravitational potential

Equations:

$$\frac{1}{a^3}\left(a^3\varrho\right)^{\cdot} + \frac{1}{a}\nabla_i\left(\varrho v^i\right) = 0, \tag{106}$$

$$\frac{1}{a}\left(av_i\right)^{\cdot} + \frac{1}{a}v^j\nabla_j v_i + \frac{1}{a\varrho}\left(\nabla_i p + \nabla_j\Pi_i^j\right) - \frac{1}{a}\nabla_i U = 0, \tag{107}$$

$$\frac{\Delta}{a^2}U + 4\pi G\left(\varrho - \varrho_b\right) = 0. \tag{108}$$

$$\left(\frac{\partial}{\partial t} + \frac{1}{a}\mathbf{v}\cdot\nabla\right)\Pi + \left(3\frac{\dot{a}}{a} + \frac{1}{a}\nabla\cdot\mathbf{v}\right)\frac{p}{\varrho} + \frac{1}{\varrho a}\left(Q^i{}_{|i} + \Pi^i_j v^j{}_{|i}\right) = 0. \tag{109}$$

Newtonian, indeed!

JCAP03 (2008) 010

First Post-Newtonian equations (without taking temporal gauge):

$$\begin{aligned}
&\frac{1}{a^3}\left(a^3\varrho\right)^{\cdot} + \frac{1}{a}\left(\varrho v^i\right)_{|i} + \frac{1}{c^2}\varrho\left(\frac{\partial}{\partial t} + \frac{1}{a}\mathbf{v}\cdot\nabla\right)\left(\frac{1}{2}v^2 + 3U\right) \\
&\qquad = -\frac{1}{c^2}\left[\frac{1}{a}\left(Q^i{}_{|i} + \Pi^i_j v^j{}_{|i}\right) + \varrho\left(\frac{\partial}{\partial t} + \frac{1}{a}\mathbf{v}\cdot\nabla\right)\Pi + \left(3\frac{\dot a}{a} + \frac{1}{a}\nabla\cdot\mathbf{v}\right)p\right],
\end{aligned} \tag{114}$$

$$\begin{aligned}
&\frac{1}{a}\left(av_i\right)^{\cdot} + \frac{1}{a}v_{i|j}v^j - \frac{1}{a}U_{,i} + \frac{1}{a}\frac{p_{,i}}{\varrho} + \frac{1}{c^2}\Bigg[-\frac{1}{a}v^2U_{,i} + \frac{2}{a}\left(U^2 - \Phi\right)_{,i} - \frac{1}{a}\left(aP_i\right)^{\cdot} - \frac{2}{a}v^jP_{[i|j]} \\
&\qquad - \frac{1}{a}\left(v^2 + 4U + \Pi + \frac{p}{\varrho}\right)\frac{p_{,i}}{\varrho} + v_i\left(\frac{\partial}{\partial t} + \frac{1}{a}\mathbf{v}\cdot\nabla\right)\left(\frac{1}{2}v^2 + 3U + \frac{p}{\varrho}\right) \\
&\qquad - \left(3\frac{\dot a}{a} + \frac{1}{a}v^j{}_{|j}\right)\frac{p}{\varrho}v_i\Bigg] \\
&\qquad = \frac{1}{a\varrho}\Pi^j_{i|j} + \frac{1}{c^2}\frac{1}{\varrho}\left\{\frac{1}{a^4}\left[a^4\left(Q_i + \Pi^j_i v_j\right)\right]^{\cdot} + \frac{1}{a}\left(Q^jv_i + Q_iV^j\right)_{|j} - \frac{4}{a}U\Pi^j_{i|j}\right\},
\end{aligned} \tag{115}$$

$$\begin{aligned}
&\frac{\Delta}{a^2}U + 4\pi G\left(\varrho - \varrho_b\right) + \frac{1}{c^2}\Bigg\{\frac{1}{a^2}\left[2\Delta\Phi - 2U\Delta U + \left(aP^i{}_{|i}\right)^{\cdot}\right] + 3\ddot U + 9\frac{\dot a}{a}\dot U + 6\frac{\ddot a}{a}U \\
&\qquad + 8\pi G\left[\varrho v^2 + \tfrac{1}{2}\left(\varrho\Pi - \varrho_b\Pi_b\right) + \tfrac{3}{2}\left(p - p_b\right)\right]\Bigg\} = 0,
\end{aligned} \tag{119}$$

$$\frac{\Delta}{a^2}P_i = -16\pi G\varrho v_i + \frac{1}{a}\left(\frac{1}{a}P^j{}_{|j} + 4\dot U + 4\frac{\dot a}{a}U\right)_{,i}. \tag{120}$$

Conclusion

- **Perturbation method:** Fully relativistic but weakly nonlinear.

  Pure relativistic third-order corrections: $$\varphi_v \sim \frac{\delta\Phi}{c^2} \sim 10^{-5}$$

- **PN approximation:** Fully nonlinear but weakly relativistic.

  PN corrections: $$\frac{GM}{Rc^2} \sim \frac{\delta\Phi}{c^2} \sim \frac{v^2}{c^2} \sim 10^{-6} - 10^{-4}$$

- Newtonian theory looks quite reliable in cosmological dynamics.
- Secular effects? Require numerical simulation.
- Equations are presented without taking the temporal gauge.
- Newtonian: action at a distance (Laplacian)
  → PN: propagation with speed of light (D'Alembertian)
- PN approximation including a scalar field as a Dark Energy?
  In progress.