Cosmological Perturbation Theory

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References

"The evolution of linear perturbations of FRW models has been discussed by a large number of authors and is very nearly a closed book."

George Efstatiou (1989)

Lecture 1

1. **History**
2. **Introduction**
3. **Covariant $(1+3)$ formulation**
4. **ADM $(3+1)$ formulation**
5. **Newtonian cosmological perturbations**

"*The theory of linear (i.e., small) perturbations of the expanding, isotropic, and homogeneous Friedmann cosmology springs into existence virtually full-grown with the work of Lifshitz (1946).*"

Press and Vishniac (1980)

1. History

- **Background:** spatially homogeneous and isotropic world model
  - Relativistic (Friedmann 1922)
  - Newtonian (Milne 1933)
- **Structures:** general linear perturbations
  - Relativistic (Lifshitz 1946)
  - Newtonian (Bonnor 1957)
  - CMB anisotropy (Sachs-Wolfe 1967)
- **Observations:**
  - Expansion (Hubble 1929)
  - CMB (Penzias-Wilson 1965)
  - 3d large scale structure (mid 80's -)
  - 2d anisotropies in CMB (COBE 1992 -)
  - 2dF, SDSS, WMAP, Planck, ...
  - Acceleration (SN teams 1998)

Evgeniĭ Mikhaĭlovich Lifshitz (1915 - 1985)

E. M. Lifshitz

2. Introduction

Gravitation:

- Newtonian gravity
  - Non-relativistic (no c)
  - No strong pressure allowed
  - No horizon
  - No gravitational wave
- Einstein gravity
  - Relativistic gravity (simplest)
- Generalized gravity
  - Brans-Dicke theory
  - Quantum corrections
  - Low energy limit of unified theories (*e.g.*, string theory)

Methods:

1. Newtonian:
   (a) Hydrodynamic equations
2. Relativistic:
   (a) Einstein's equation (Lifshitz 1946)
   (b) Covariant equations (Fluid-like, $\tilde{u}_a$; Hawking 1966),
   (c) ADM equations ($3+1$, $\tilde{n}_a$; Bardeen 1980, 1988)
   (d) Action (Lukash 1980; Mukhanov 1988)
3. Energy-momentum content:
   (a) Hydrodynamic fluids
   (b) Scalar fields

Three modes:

1. Scalar-type: Density condensation
2. Vector-type: Rotation
3. Tensor-type: Gravitational wave

- **Decouple** to the linear-order in spatially homogeneous-isotropic background.
  - In anisotropic background world model (e.g., Bianchi type I) the non-vanishing shear in the background will **couple** all three-types of perturbations.
  - To the second order in perturbations the linear order perturbations of all three-types will **source** (thus, couple) three-types of perturbation in the second order.

Classical Evolution:

1. Scalar-type: super-sound-horizon scale conservation
2. Rotation: angular momentum conservation
3. Gravitational wave: super-horizon scale conservation

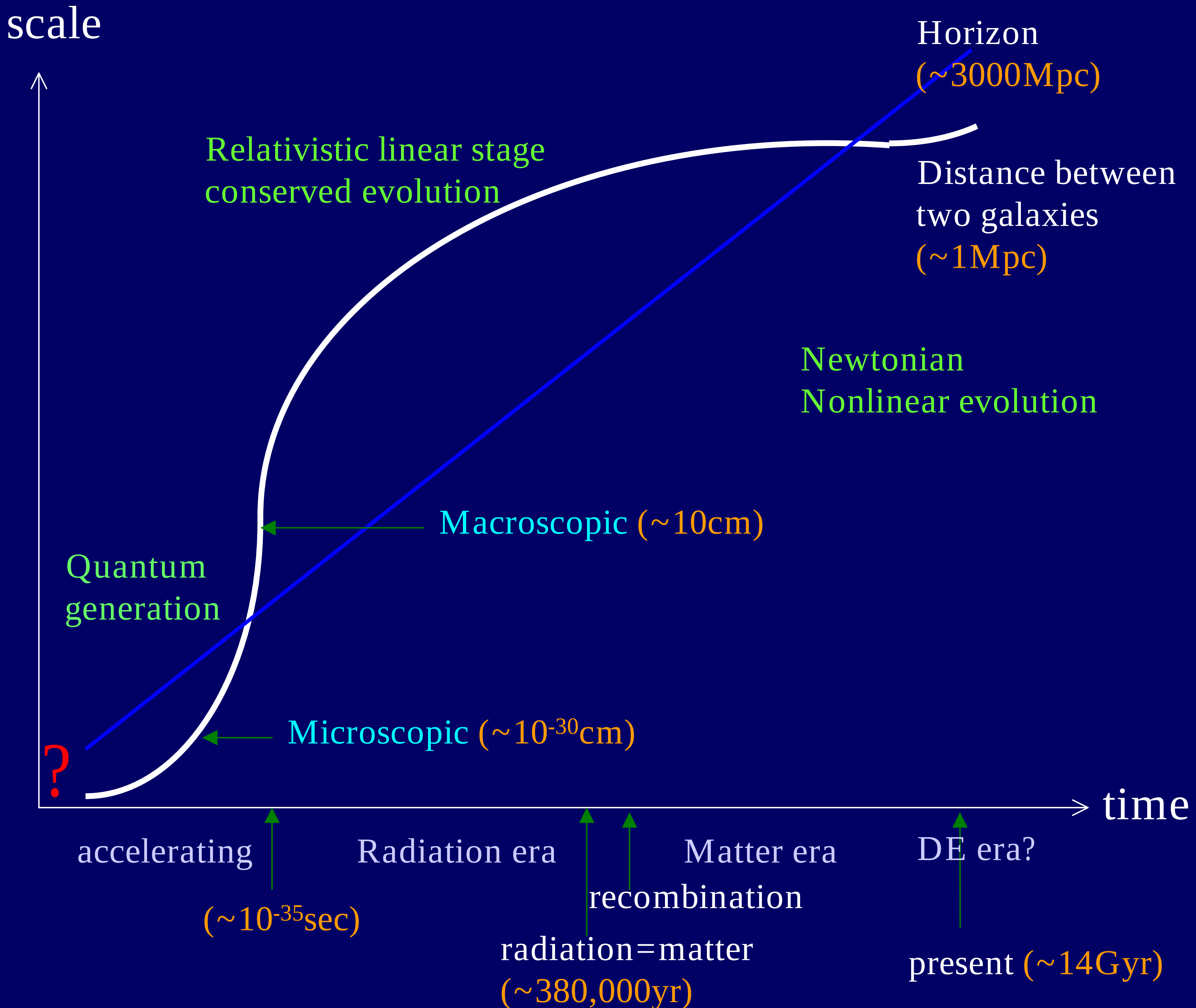
- (1,2): independently of horizon crossing.
- (1-3): independently of changing equation of state, changing potential, changing gravity theories.

Origin:

1. Topological defects
2. Quantum fluctuations + inflation

Three stages:

1. Quantum generation (becomes macroscopic by inflation)
2. Classical evolution (super-sound-horizon, linear, conserved)
3. Nonlinear evolution (far inside horizon, Newtonian simulation)


scale
time
Horizon
(~3000Mpc)
Relativistic linear stage
conserved evolution
Distance between
two galaxies
(~1Mpc)
Newtonian
Nonlinear evolution
Macroscopic (~10cm)
Quantum
generation
Microscopic (~10^-30cm)
?
accelerating
Radiation era
Matter era
DE era?
(~10^-35sec)
recombination
radiation=matter
(~380,000yr)
present (~14Gyr)

Einstein's equation

Action:

$$\tilde{S} = \int \left[\frac{1}{16\pi G}\left(\tilde{R} - 2\Lambda\right) + \tilde{L}_m\right]\sqrt{-\tilde{g}}d^4x, \tag{1}$$

where $\tilde{g} \equiv \det(\tilde{g}_{ab})$ and $\delta(\sqrt{-\tilde{g}}\tilde{L}_m) \equiv \frac{1}{2}\sqrt{-\tilde{g}}\tilde{T}^{ab}\delta\tilde{g}_{ab}$.

Einstein's equation:

$$\tilde{G}_{ab} = 8\pi G\tilde{T}_{ab} - \Lambda\tilde{g}_{ab}. \tag{2}$$

Energy-momentum conservation:

$$\tilde{T}^b_{a;b} = 0. \tag{3}$$

Latin indices $a, b, c, \cdots =$ spacetime; Greek indices $\alpha, \beta, \gamma, \cdots =$ space.
Tildes indicate covariant quantities.
Signature convention: $(-1, +1, +1, +1)$.
We set $c \equiv 1 \equiv \hbar$.
Curvature convention:

$$\tilde{u}_{a;bc} - \tilde{u}_{a;cb} \equiv \tilde{u}_d\tilde{R}^d{}_{abc}, \tag{4}$$

$$\tilde{R}^a{}_{bcd} \equiv \tilde{\Gamma}^a_{bd,c} - \tilde{\Gamma}^a_{bc,d} + \tilde{\Gamma}^e_{bd}\tilde{\Gamma}^a_{ce} - \tilde{\Gamma}^e_{bc}\tilde{\Gamma}^a_{de},$$

$$\tilde{R}_{ab} \equiv \tilde{R}^c{}_{acb}, \quad \tilde{R} \equiv \tilde{R}^a_a, \quad \tilde{G}_{ab} \equiv \tilde{R}_{ab} - \frac{1}{2}\tilde{R}\tilde{g}_{ab},$$

$$\tilde{\Gamma}^a_{bc} \equiv \frac{1}{2}\tilde{g}^{ad}\left(\tilde{g}_{bd,c} + \tilde{g}_{dc,b} - \tilde{g}_{bc,d}\right). \tag{5}$$

3. Covariant $(1+3)$ Formulation

See Ehlers (1961), Ellis (1971,1973) [8, 35, 51].

Introduce time-like normalized ($\tilde{u}^a\tilde{u}_a \equiv -1$) four-vector field $\tilde{u}_a$ in every spacetime point.
The projection tensor:

$$\tilde{h}_{ab} \equiv \tilde{g}_{ab} + \tilde{u}_a\tilde{u}_b; \quad \tilde{h}_{ab}\tilde{u}^b = 0, \quad \tilde{h}^a_a = 3. \tag{6}$$

Decomposition:

$$\begin{aligned}
&\tilde{h}^c_a\tilde{h}^d_b\tilde{u}_{c;d} = \tilde{h}^c_{[a}\tilde{h}^d_{b]}\tilde{u}_{c;d} + \tilde{h}^c_{(a}\tilde{h}^d_{b)}\tilde{u}_{c;d} \equiv \tilde{\omega}_{ab} + \tilde{\theta}_{ab} = \tilde{u}_{a;b} + \tilde{a}_a\tilde{u}_b, \\
&\tilde{\sigma}_{ab} \equiv \tilde{\theta}_{ab} - \frac{1}{3}\tilde{\theta}\tilde{h}_{ab}, \quad \tilde{\theta} \equiv \tilde{u}^a{}_{;a}, \quad \tilde{a}_a \equiv \dot{\tilde{u}}_a \equiv \tilde{u}_{a;b}\tilde{u}^b, \\
&\tilde{u}_{a;b} = \tilde{\omega}_{ab} + \tilde{\sigma}_{ab} + \frac{1}{3}\tilde{\theta}\tilde{h}_{ab} - \tilde{a}_a\tilde{u}_b.
\end{aligned} \tag{7}$$

$\tilde{\theta}$ = expansion scalar
$\tilde{a}_a$ = acceleration vector
$\tilde{\omega}_{ab}$ = rotation tensor
$\tilde{\sigma}_{ab}$ = shear tensor

Introduce

$$\tilde{\omega}^a \equiv \frac{1}{2}\tilde{\eta}^{abcd}\tilde{u}_b\tilde{\omega}_{cd}, \quad \tilde{\omega}_{ab} = \tilde{\eta}_{abcd}\tilde{\omega}^c\tilde{u}^d, \quad \tilde{\omega}^2 \equiv \frac{1}{2}\tilde{\omega}^{ab}\tilde{\omega}_{ab} = \tilde{\omega}^a\tilde{\omega}_a, \quad \tilde{\sigma}^2 \equiv \frac{1}{2}\tilde{\sigma}^{ab}\tilde{\sigma}_{ab}, \tag{8}$$

where $\tilde{\eta}^{abcd} = \tilde{\eta}^{[abcd]}$ with $\tilde{\eta}^{1234} = 1/\sqrt{-\tilde{g}}$.
Indices surrounded by ( ) and [ ] are the symmetrization and anti-symmetrization symbols.

The energy-momentum tensor is decomposed into fluid quantities based on $\tilde{u}^a$:

$$\tilde{T}_{ab} \equiv \tilde{\mu}\tilde{u}_a\tilde{u}_b + \tilde{p}\left(\tilde{g}_{ab} + \tilde{u}_a\tilde{u}_b\right) + \tilde{q}_a\tilde{u}_b + \tilde{q}_b\tilde{u}_a + \tilde{\pi}_{ab}, \tag{9}$$

where

$$\tilde{u}^a\tilde{q}_a \equiv 0 \equiv \tilde{u}^a\tilde{\pi}_{ab}, \quad \tilde{\pi}_{ab} \equiv \tilde{\pi}_{ba}, \quad \tilde{\pi}^a_a \equiv 0. \tag{10}$$

Thus

$$\tilde{\mu} \equiv \tilde{T}_{ab}\tilde{u}^a\tilde{u}^b, \quad \tilde{p} \equiv \frac{1}{3}\tilde{T}_{ab}\tilde{h}^{ab}, \quad \tilde{q}_a \equiv -\tilde{T}_{cd}\tilde{u}^c\tilde{h}^d_a, \quad \tilde{\pi}_{ab} \equiv \tilde{T}_{cd}\tilde{h}^c_a\tilde{h}^d_b - \tilde{p}\tilde{h}_{ab}. \tag{11}$$

$\tilde{\mu}$ = energy density
$\tilde{p}$ = (isotropic) pressure
$\tilde{q}_a$ = flux vector
$\tilde{\pi}_{ab}$ = anisotropic pressure (stress)

Frame choice:

We have $\tilde{\mu}$ (1-component), $\tilde{p}$ (1), $\tilde{q}_a$ (3), $\tilde{\pi}_{ab}$ (5), and $\tilde{u}_a$ (3), thus total 13 components to fix $\tilde{T}_{ab}$ which has only 10 independent components.
Thus, we have freedom to choose frame vector:

Energy-frame:	$\tilde{q}_a \equiv 0$
Normal-frame:	$\tilde{u}_a \equiv \tilde{n}_a$ with $\tilde{n}_\alpha \equiv 0$

Covariant equations:

Raychaudhuri equation:

$$\dot{\tilde{\theta}} + \frac{1}{3}\tilde{\theta}^2 - \tilde{a}^a{}_{;a} + 2\left(\tilde{\sigma}^2 - \tilde{\omega}^2\right) + 4\pi G\left(\tilde{\mu} + 3\tilde{p}\right) - \Lambda = 0. \tag{12}$$

Vorticity propagation:

$$\tilde{h}^a_b \dot{\tilde{\omega}}^b + \frac{2}{3}\tilde{\theta}\tilde{\omega}^a = \tilde{\sigma}^a_b\tilde{\omega}^b + \frac{1}{2}\tilde{\eta}^{abcd}\tilde{u}_b\tilde{a}_{c;d}. \tag{13}$$

Shear propagation:

$$\tilde{h}^c_a\tilde{h}^d_b\left(\dot{\tilde{\sigma}}_{cd} - a_{(c;d)}\right) - \tilde{a}_a\tilde{a}_b + \tilde{\omega}_a\tilde{\omega}_b + \tilde{\sigma}_{ac}\tilde{\sigma}^c_b + \frac{2}{3}\tilde{\theta}\tilde{\sigma}_{ab} - \frac{1}{3}\tilde{h}_{ab}\left(\tilde{\omega}^2 + 2\tilde{\sigma}^2 - \tilde{a}^c{}_{;c}\right)$$
$$+\tilde{E}_{ab} - 4\pi G\tilde{\pi}_{ab} = 0. \tag{14}$$

Three constraint equations:

$$\tilde{h}_{ab}\left(\tilde{\omega}^{bc}{}_{;c} - \tilde{\sigma}^{bc}{}_{;c} + \frac{2}{3}\tilde{\theta}^{;b}\right) + \left(\tilde{\omega}_{ab} + \tilde{\sigma}_{ab}\right)\tilde{a}^b = 8\pi G\tilde{q}_a, \tag{15}$$

$$\tilde{\omega}^a{}_{;a} = 2\tilde{\omega}^b\tilde{a}_b, \tag{16}$$

$$\tilde{H}_{ab} = 2\tilde{a}_{(a}\tilde{\omega}_{b)} - \tilde{h}^c_a\tilde{h}^d_b\left(\tilde{\omega}_{(c}{}^{e;f} + \tilde{\sigma}_{(c}{}^{e;f}\right)\tilde{\eta}_{d)gef}\tilde{u}^g. \tag{17}$$

The energy and the momentum conservation equations:

$$\dot{\tilde{\mu}} + \left(\tilde{\mu} + \tilde{p}\right)\tilde{\theta} + \tilde{\pi}^{ab}\tilde{\sigma}_{ab} + \tilde{q}^a{}_{;a} + \tilde{q}^a\tilde{a}_a = 0, \tag{18}$$

$$\left(\tilde{\mu} + \tilde{p}\right)\tilde{a}_a + \tilde{h}^b_a\left(\tilde{p}_{,b} + \tilde{\pi}^c_{b;c} + \dot{\tilde{q}}_b\right) + \left(\tilde{\omega}_{ab} + \tilde{\sigma}_{ab} + \frac{4}{3}\tilde{\theta}\tilde{h}_{ab}\right)\tilde{q}^b = 0. \tag{19}$$

In the normal-frame, thus $\tilde{u}_a = \tilde{n}_a$ with $\tilde{n}_\alpha \equiv 0$, we have $\tilde{\omega}_{ab} = 0$.
The trace part of the Gauss equation give:

$$\boxed{\tilde{R}^{(3)} = 2\left(-\frac{1}{3}\tilde{\theta}^2 + \tilde{\sigma}^2 + 8\pi G\tilde{\mu} + \Lambda\right).} \tag{20}$$

$\tilde{R}^{(3)}$ is the scalar curvature of the hypersurface normal to $\tilde{n}_a$.

Weyl parts:

The Weyl (conformal) curvature:

$$\tilde{C}_{abcd} \equiv \tilde{R}_{abcd} - \frac{1}{2}\left(\tilde{g}_{ac}\tilde{R}_{bd} + \tilde{g}_{bd}\tilde{R}_{ac} - \tilde{g}_{bc}\tilde{R}_{ad} - \tilde{g}_{ad}\tilde{R}_{bc}\right) + \frac{1}{6}\tilde{R}\left(\tilde{g}_{ac}\tilde{g}_{bd} - \tilde{g}_{ad}\tilde{g}_{bc}\right). \tag{21}$$

The electric and magnetic parts of the Weyl curvature:

$$\tilde{E}_{ab} \equiv \tilde{C}_{acbd}\tilde{u}^c\tilde{u}^d, \quad \tilde{H}_{ab} \equiv \frac{1}{2}\tilde{\eta}_{ac}{}^{ef}\tilde{C}_{efbd}\tilde{u}^c\tilde{u}^d. \tag{22}$$

Four quasi-Maxwellian equations:

div$E$:

$$\tilde{h}^a_b \tilde{h}^c_d \tilde{E}^{bd}{}_{;c} - \tilde{\eta}^{abcd} \tilde{u}_b \tilde{\sigma}^e_c \tilde{H}_{de} + 3\tilde{H}^a_b \tilde{\omega}^b$$
$$= 4\pi G \Big(\frac{2}{3} \tilde{h}^{ab} \tilde{\mu}_{,b} - \tilde{h}^a_b \tilde{\pi}^{bc}{}_{;c} - 3\tilde{\omega}^a{}_b \tilde{q}^b + \tilde{\sigma}^a_b \tilde{q}^b + \tilde{\pi}^a_b \tilde{a}^b - \frac{2}{3} \tilde{\theta} \tilde{q}^a \Big). \tag{23}$$

div$H$:

$$\tilde{h}^a_b \tilde{h}^c_d \tilde{H}^{bd}{}_{;c} + \tilde{\eta}^{abcd} \tilde{u}_b \tilde{\sigma}^e_c \tilde{E}_{de} - 3\tilde{E}^a_b \tilde{\omega}^b$$
$$= 4\pi G \Big\{ 2\left(\tilde{\mu} + \tilde{p}\right) \tilde{\omega}^a + \tilde{\eta}^{abcd} \tilde{u}_b \left[\tilde{q}_{c;d} + \tilde{\pi}_{ce} \left(\tilde{\omega}^e{}_d + \tilde{\sigma}^e{}_d \right) \right] \Big\}. \tag{24}$$

$\dot{E}$:

$$\tilde{h}^a_c \tilde{h}^b_d \dot{\tilde{\tilde{E}}}^{cd} + \left(\tilde{H}^f_{d;e} \tilde{h}^{(a}_f - 2\tilde{a}_d \tilde{H}^{(a}_e \right) \tilde{\eta}^{b)cde} \tilde{u}_c + \tilde{h}^{ab} \tilde{\sigma}^{cd} \tilde{E}_{cd} + \tilde{\theta} \tilde{E}^{ab} - \tilde{E}^{(a}_c \left(3\tilde{\sigma}^{b)c} + \tilde{\omega}^{b)c} \right)$$
$$= 4\pi G \Big[- \left(\tilde{\mu} + \tilde{p}\right) \tilde{\sigma}^{ab} - 2\tilde{a}^{(a} \tilde{q}^{b)} - \tilde{h}^{(a}_c \tilde{h}^{b)}_d \left(\tilde{q}^{c;d} + \dot{\tilde{\tilde{\pi}}}^{cd} \right) - \left(\tilde{\omega}_c{}^{(a} + \tilde{\sigma}^{(a}_c \right) \tilde{\pi}^{b)c}$$
$$- \frac{1}{3} \tilde{\theta} \tilde{\pi}^{ab} + \frac{1}{3} \left(\tilde{q}^c{}_{;c} + \tilde{a}_c \tilde{q}^c + \tilde{\pi}^{cd} \tilde{\sigma}_{cd} \right) \tilde{h}^{ab} \Big]. \tag{25}$$

$\dot{H}$:

$$\tilde{h}^a_c \tilde{h}^b_d \dot{\tilde{\tilde{H}}}^{cd} - \left(\tilde{E}^f_{d;e} \tilde{h}^{(a}_f - 2\tilde{a}_d \tilde{E}^{(a}_e \right) \tilde{\eta}^{b)cde} \tilde{u}_c + \tilde{h}^{ab} \tilde{\sigma}^{cd} \tilde{H}_{cd} + \tilde{\theta} \tilde{H}^{ab} - \tilde{H}^{(a}_c \left(3\tilde{\sigma}^{b)c} + \tilde{\omega}^{b)c} \right)$$
$$= 4\pi G \left[\left(\tilde{q}_e \tilde{\sigma}^{(a}_d - \tilde{\pi}^f_{d;e} \tilde{h}^{(a}_f \right) \tilde{\eta}^{b)cde} \tilde{u}_c + \tilde{h}^{ab} \tilde{\omega}_c \tilde{q}^c - 3\tilde{\omega}^{(a} \tilde{q}^{b)} \right]. \tag{26}$$

4. ADM (3 + 1) Formulation

Arnowitt-Deser-Misner (1962), Bardeen (1980, 1988) [3, 4, 5, 51].

The metric:

$$ds^2 = -N^2(dx^0)^2 + h_{\alpha\beta}\left(dx^\alpha + N^\alpha dx^0\right)\left(dx^\beta + N^\beta dx^0\right). \tag{27}$$

N = lapse function

N_α = shift vector

$h_{\alpha\beta}$ = three-space metric

$h^{\alpha\beta}$ = inverse three-space metric, $h^{\alpha\beta}h_{\beta\gamma} \equiv \delta^\alpha_\gamma$

ADM variables are based on $h_{\alpha\beta}$ as the metric.

Metric and inverse metric tensors:

$$\tilde{g}_{00} \equiv -N^2 + N^\alpha N_\alpha, \quad \tilde{g}_{0\alpha} \equiv N_\alpha, \quad \tilde{g}_{\alpha\beta} \equiv h_{\alpha\beta}, \tag{28}$$

$$\tilde{g}^{00} = -N^{-2}, \quad \tilde{g}^{0\alpha} = N^{-2}N^\alpha, \quad \tilde{g}^{\alpha\beta} = h^{\alpha\beta} - N^{-2}N^\alpha N^\beta, \tag{29}$$

The extrinsic curvature:

$$K_{\alpha\beta} \equiv \frac{1}{2N}\left(N_{\alpha:\beta} + N_{\beta:\alpha} - h_{\alpha\beta,0}\right); \quad K \equiv h^{\alpha\beta}K_{\alpha\beta}, \quad \bar{K}_{\alpha\beta} \equiv K_{\alpha\beta} - \frac{1}{3}h_{\alpha\beta}K, \tag{30}$$

The intrinsic curvatures based on $h_{\alpha\beta}$:

$$R^{(h)\alpha}{}_{\beta\gamma\delta} \equiv \Gamma^{(h)\alpha}{}_{\beta\delta,\gamma} - \Gamma^{(h)\alpha}{}_{\beta\gamma,\delta} + \Gamma^{(h)\epsilon}{}_{\beta\delta}\Gamma^{(h)\alpha}{}_{\gamma\epsilon} - \Gamma^{(h)\epsilon}{}_{\beta\gamma}\Gamma^{(h)\alpha}{}_{\delta\epsilon},$$

$$R^{(h)}_{\alpha\beta} \equiv R^{(h)\gamma}{}_{\alpha\gamma\beta}, \quad R^{(h)} \equiv h^{\alpha\beta}R^{(h)}_{\alpha\beta}, \quad \bar{R}^{(h)}_{\alpha\beta} \equiv R^{(h)}_{\alpha\beta} - \frac{1}{3}h_{\alpha\beta}R^{(h)},$$

$$\Gamma^{(h)\alpha}{}_{\beta\gamma} \equiv \frac{1}{2}h^{\alpha\delta}\left(h_{\beta\delta,\gamma} + h_{\delta\gamma,\beta} - h_{\beta\gamma,\delta}\right). \tag{31}$$

The normal vector $\tilde{n}_a$:

$$\tilde{n}_0 \equiv -N, \quad \tilde{n}_\alpha \equiv 0; \quad \tilde{n}^0 = N^{-1}, \quad \tilde{n}^\alpha = -N^{-1} N^\alpha. \tag{32}$$

The fluid quantities:

$$E \equiv \tilde{n}_a \tilde{n}_b \tilde{T}^{ab}, \quad J_\alpha \equiv -\tilde{n}_b \tilde{T}^b_\alpha, \quad S_{\alpha\beta} \equiv \tilde{T}_{\alpha\beta}; \quad S \equiv h^{\alpha\beta} S_{\alpha\beta}, \quad \bar{S}_{\alpha\beta} \equiv S_{\alpha\beta} - \frac{1}{3} h_{\alpha\beta} S. \tag{33}$$

E = energy density
J_α = flux vector
S = 3×pressure
$\bar{S}_{\alpha\beta}$ = anisotropic stress

The ADM fluid quantities correspond to the covariant fluid quantities based on the normal-frame vector:

$$E = \tilde{\mu}, \quad S = 3\tilde{p}, \quad J_\alpha = \tilde{q}_\alpha, \quad \bar{S}_{\alpha\beta} = \tilde{\pi}_{\alpha\beta}. \tag{34}$$

ADM equations:

Energy constraint equation:

$$R^{(h)} = \bar{K}^{\alpha\beta}\bar{K}_{\alpha\beta} - \frac{2}{3}K^2 + 16\pi G E + 2\Lambda. \tag{35}$$

Momentum constraint equation:

$$\bar{K}^{\beta}_{\alpha:\beta} - \frac{2}{3}K_{,\alpha} = 8\pi G J_{\alpha}. \tag{36}$$

Trace of ADM propagation equation:

$$K_{,0}N^{-1} - K_{,\alpha}N^{\alpha}N^{-1} + N^{:\alpha}{}_{\alpha}N^{-1} - \bar{K}^{\alpha\beta}\bar{K}_{\alpha\beta} - \frac{1}{3}K^2 - 4\pi G\left(E + S\right) + \Lambda = 0. \tag{37}$$

Tracefree ADM propagation equation:

$$\begin{aligned}&\bar{K}^{\alpha}_{\beta,0}N^{-1} - \bar{K}^{\alpha}_{\beta:\gamma}N^{\gamma}N^{-1} + \bar{K}_{\beta\gamma}N^{\alpha:\gamma}N^{-1} - \bar{K}^{\alpha}_{\gamma}N^{\gamma}{}_{:\beta}N^{-1} \\ &\quad = K\bar{K}^{\alpha}_{\beta} - \left(N^{:\alpha}{}_{\beta} - \frac{1}{3}\delta^{\alpha}_{\beta}N^{:\gamma}{}_{\gamma}\right)N^{-1} + \bar{R}^{(h)\alpha}{}_{\beta} - 8\pi G\bar{S}^{\alpha}_{\beta}.\end{aligned} \tag{38}$$

Energy conservation equation:

$$E_{,0}N^{-1} - E_{,\alpha}N^{\alpha}N^{-1} - K\left(E + \frac{1}{3}S\right) - \bar{S}^{\alpha\beta}\bar{K}_{\alpha\beta} + N^{-2}\left(N^2 J^{\alpha}\right)_{:\alpha} = 0. \tag{39}$$

Momentum conservation equation:

$$J_{\alpha,0}N^{-1} - J_{\alpha:\beta}N^{\beta}N^{-1} - J_{\beta}N^{\beta}{}_{:\alpha}N^{-1} - KJ_{\alpha} + EN_{,\alpha}N^{-1} + S^{\beta}_{\alpha:\beta} + S^{\beta}_{\alpha}N_{,\beta}N^{-1} = 0. \tag{40}$$

Multi-component case

In the multi-component situation:

$$\tilde{T}_{ab} = \sum_k \tilde{T}_{(k)ab}; \quad \tilde{T}_{(i)a;b}^{\ \ \ \ \ b} \equiv \tilde{I}_{(i)a}, \quad \sum_k \tilde{I}_{(k)a} \equiv 0. \tag{41}$$

Based on the normal-frame vector:

$$\tilde{\mu} = \sum_k \tilde{\mu}_{(k)}, \quad \tilde{p} = \sum_k \tilde{p}_{(k)}, \quad \tilde{q}_a = \sum_k \tilde{q}_{(k)a}, \quad \tilde{\pi}_{ab} = \sum_k \tilde{\pi}_{(k)ab}. \tag{42}$$

In ADM notation $E = \tilde{\mu}$, $S = 3\tilde{p}$, $J_\alpha = \tilde{q}_\alpha$, $\bar{S}_{\alpha\beta} = \tilde{\pi}_{\alpha\beta}$, thus

$$E = \sum_k E_{(k)}, \quad S = \sum_k S_{(k)}, \quad J_\alpha = \sum_k J_{(k)\alpha}, \quad \bar{S}_{\alpha\beta} = \sum_k \bar{S}_{(k)\alpha\beta}. \tag{43}$$

(41) gives the energy and momentum conservation equations for individual component:

$$E_{(i),0}N^{-1} - E_{(i),\alpha}N^\alpha N^{-1} - K\left(E_{(i)} + \frac{1}{3}S_{(i)}\right) - \bar{S}^{\alpha\beta}_{(i)}\bar{K}_{\alpha\beta} + N^{-2}\left(N^2 J^\alpha_{(i)}\right)_{:\alpha} = -\frac{1}{N}\left(\tilde{I}_{(i)0} - \tilde{I}_{(i)\alpha}N^\alpha\right), \tag{44}$$

$$J_{(i)\alpha,0}N^{-1} - J_{(i)\alpha:\beta}N^\beta N^{-1} - J_{(i)\beta}N^\beta_{\ :\alpha}N^{-1} - KJ_{(i)\alpha} + E_{(i)}N_{,\alpha}N^{-1} + S_{(i)\alpha:\beta}^{\ \ \ \ \ \beta} + S_{(i)\alpha}^{\ \ \ \ \beta}N_{,\beta}N^{-1} = \tilde{I}_{(i)\alpha}. \tag{45}$$

(35-40,44,45) provide a complete set of equations.

5. Newtonian Cosmological Perturbations

Hydrodynamic equations:

Continuity (mass conservation), Euler (momentum conservation), and Poisson's equations:

$$\dot{\varrho} + \nabla \cdot (\varrho \mathbf{v}) = 0, \tag{46}$$

$$\dot{\mathbf{v}} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\varrho} \nabla p - \nabla \Phi, \tag{47}$$

$$\nabla^2 \Phi = 4\pi G \varrho. \tag{48}$$

Uniform background:

$\mathbf{v} = H\mathbf{r}$ where $H \equiv \frac{\dot{a}}{a}$.
(46-48) give:

$$\dot{\varrho} + 3H\varrho = 0, \tag{49}$$

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\varrho. \tag{50}$$

$$\Phi = \frac{2\pi G}{3}\varrho \mathbf{r}^2. \tag{51}$$

From these we have

$$H^2 = \frac{8\pi G}{3}\varrho + \frac{2E}{a^2}, \tag{52}$$

where E is an integration constant.

Perturbations:

Introduce perturbations:

$$\varrho = \bar{\varrho} + \delta\varrho \equiv \bar{\varrho}\,(1+\delta)\,, \quad p = \bar{p} + \delta p, \quad \mathbf{v} = H\mathbf{r} + \mathbf{u}, \quad \Phi = \bar{\Phi} + \delta\Phi. \tag{53}$$

Perturbed parts of (46-48) give:

$$\frac{\partial}{\partial t}\delta\varrho + H\mathbf{r}\cdot\nabla\delta\varrho + 3H\delta\varrho + \bar{\varrho}\nabla\cdot\mathbf{u} + \nabla\cdot(\delta\varrho\mathbf{u}) = 0, \tag{54}$$

$$\frac{\partial}{\partial t}\mathbf{u} + H\mathbf{r}\cdot\nabla\mathbf{u} + H\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{\nabla\delta p}{\bar{\varrho}+\delta\varrho} - \nabla\delta\Phi, \tag{55}$$

$$\nabla^2\delta\Phi = 4\pi G\delta\varrho. \tag{56}$$

Introduce the comoving coordinate $\mathbf{x}$

$$\mathbf{r} \equiv a(t)\mathbf{x}, \tag{57}$$

thus

$$\nabla = \nabla_{\mathbf{r}} = \frac{1}{a}\nabla_{\mathbf{x}},$$

$$\frac{\partial}{\partial t} = \left.\frac{\partial}{\partial t}\right|_{\mathbf{r}} = \left.\frac{\partial}{\partial t}\right|_{\mathbf{x}} + \left(\left.\frac{\partial}{\partial t}\right|_{\mathbf{r}}\mathbf{x}\right)\cdot\nabla_{\mathbf{x}} = \left.\frac{\partial}{\partial t}\right|_{\mathbf{x}} - H\mathbf{x}\cdot\nabla_{\mathbf{x}}. \tag{58}$$

Neglecting the subindex $\mathbf{x}$, we have

$$\dot{\delta} + \frac{1}{a}\nabla\cdot\mathbf{u} = -\frac{1}{a}\nabla\cdot(\delta\mathbf{u})\,, \tag{59}$$

$$\dot{\mathbf{u}} + H\mathbf{u} + \frac{1}{a}\nabla\delta\Phi = -\frac{1}{a\bar{\varrho}}\frac{\nabla\delta p}{1+\delta} - \frac{1}{a}\mathbf{u}\cdot\nabla\mathbf{u}, \tag{60}$$

$$\frac{1}{a^2}\nabla^2\delta\Phi = 4\pi G\bar{\varrho}\delta. \tag{61}$$

We introduce

$$\theta \equiv \frac{1}{a}\nabla \cdot \mathbf{u}, \quad \vec{\omega} \equiv \frac{1}{a}\nabla \times \mathbf{u}. \tag{62}$$

By applying $\frac{1}{a}\nabla\cdot$ and $\frac{1}{a}\nabla\times$ on (60) we have:

$$\dot{\theta} + 2H\theta + 4\pi G\bar{\varrho}\delta = -\frac{1}{a^2\bar{\varrho}}\nabla \cdot \left(\frac{\nabla\delta p}{1+\delta}\right) - \frac{1}{a^2}\nabla \cdot (\mathbf{u} \cdot \nabla\mathbf{u}), \tag{63}$$

$$\dot{\vec{\omega}} + 2H\vec{\omega} = \frac{1}{a^2\bar{\varrho}}\frac{(\nabla\delta) \times \nabla\delta p}{(1+\delta)^2} - \frac{1}{a^2}\nabla \times (\mathbf{u} \cdot \nabla\mathbf{u}). \tag{64}$$

Combining (59,61,63)

$$\boxed{\ddot{\delta} + 2H\dot{\delta} - 4\pi G\bar{\varrho}\delta = \frac{1}{a^2\bar{\varrho}}\nabla \cdot \left(\frac{\nabla \cdot \delta p}{1+\delta}\right) - \frac{1}{a^2}\left[a\nabla \cdot (\delta\mathbf{u})\right]^{\cdot} + \frac{1}{a^2}\nabla \cdot (\mathbf{u} \cdot \nabla\mathbf{u}).} \tag{65}$$

(59-65) are valid to **fully nonlinear** order.

To the linear order, using $\delta p \equiv v_s^2 \delta \varrho$

$$\ddot{\delta} + 2H\dot{\delta} - \Big(\underbrace{4\pi G\bar{\varrho}}_{\text{gravity}} + \underbrace{v_s^2 \frac{\Delta}{a^2}}_{\text{pressure}}\Big)\delta = 0. \tag{66}$$

Expanding in a Fourier series $\delta \propto e^{i\mathbf{k}\cdot\mathbf{x}}$ where $\mathbf{k}$ is the comoving wavevector, Jeans criteria becomes

$$\lambda_J \equiv \frac{2\pi a}{k_J} = v_s \sqrt{\frac{\pi}{G\bar{\varrho}}}. \tag{67}$$

$\lambda > \lambda_J$	$\rightarrow$	gravity wins	$\rightarrow$	grow + decay
$\lambda < \lambda_J$	$\rightarrow$	pressure gradient wins	$\rightarrow$	oscillate

> *"But if the matter was evenly disposed throughout an infinite space, it could never convene into one mass; but some of it would convene into one mass and some into another, so as to make an infinite number of great masses, scattered at great distances from one to another throughout all that infinite space."*
>
> Isaac Newton (1692)

Lecture 2

1. **Perturbed world model**
2. **Basic equations**
3. **Gauge issue**
4. **Hydrodynamic perturbations**
5. **Covariant approach**

"Do I dare disturb the universe?"

T. S. Eliot (1888-1965)

1. Perturbed World Model

Background metric:

Spatially homogeneous and isotropic Robertson-Walker metric:

$$ds^2 = a^2 \left[-d\eta^2 + g^{(3)}_{\alpha\beta} dx^\alpha dx^\beta \right]. \tag{68}$$

$\eta =$ conformal time, $cdt \equiv a d\eta$.
$a(\eta) =$ cosmic scale factor.

Several representations:

$$ds^2 = a^2 \left[-d\eta^2 + \frac{dr^2}{1 - Kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right], \tag{69}$$

$$ds^2 = a^2 \left[-d\eta^2 + \frac{1}{\left(1 + \frac{K}{4}\bar{r}^2\right)^2} \left(dx^2 + dy^2 + dz^2 \right) \right], \tag{70}$$

$$ds^2 = a^2 \left[-d\eta^2 + d\chi^2 + \left(\frac{1}{\sqrt{K}} \sin\left(\sqrt{K}\chi\right) \right)^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right], \tag{71}$$

where

$$r \equiv \frac{\bar{r}}{1 + \frac{K}{4}\bar{r}^2}, \quad \bar{r} \equiv \sqrt{x^2 + y^2 + z^2}, \quad \chi \equiv \int^r \frac{dr}{\sqrt{1 - Kr^2}}. \tag{72}$$

Three cases depending on the sign of the spatial curvature, K.

Perturbed metric:

Introduce perturbations:

$$ds^2 = -a^2(1+2A)\,d\eta^2 - 2a^2 B_\alpha d\eta dx^\alpha + a^2\left(g^{(3)}_{\alpha\beta} + 2C_{\alpha\beta}\right)dx^\alpha dx^\beta. \tag{73}$$

Decomposition:

$$\begin{aligned} &A \equiv \alpha, \\ &B_\alpha \equiv \beta_{,\alpha} + B^{(v)}_\alpha, \\ &C_{\alpha\beta} \equiv \varphi g^{(3)}_{\alpha\beta} + \gamma_{,\alpha|\beta} + C^{(v)}_{(\alpha|\beta)} + C^{(t)}_{\alpha\beta}. \end{aligned} \tag{74}$$

B_α and $C_{\alpha\beta}$ etc, and vertical bar are based on $g^{(3)}_{\alpha\beta}$.

Scalar-type:	α, β, γ, φ			4 (2)
Vector-type:	$B^{(v)}_\alpha$, $C^{(v)}_\alpha$	transverse	$B^{(v)\alpha}{}_{\mid\alpha} \equiv 0 \equiv C^{(v)\alpha}{}_{\mid\alpha}$	4 (2)
Tensor-type:	$C^{(t)}_{\alpha\beta}$	transverse-tracefree	$C^{(t)\beta}{}_{\alpha\mid\beta} \equiv 0 \equiv C^{(t)\alpha}{}_\alpha$	2

The (scalar, vector, tensor)-type perturbation has (4, 4, 2) independent components and (2, 2, 0) components are affected by the coordinate transformation.

Linear perturbation assumes all perturbation variables are small. Thus, ignore any quadratic and higher-order combination of perturbation variables.

To linear order the types of perturbation **decouple** in the homogeneous-isotropic background.

Why linear theory?:

1. The CMB temperature and polarization anisotropies are very small $\frac{\delta T}{T} \sim 10^{-5}$.
2. The large-scale clustering of galaxies are approximately linear as the scale becomes large.

   Our own homogeneous and isotropic background world model relies on this assumption.

   Observations are not inconsistent with the assumption.

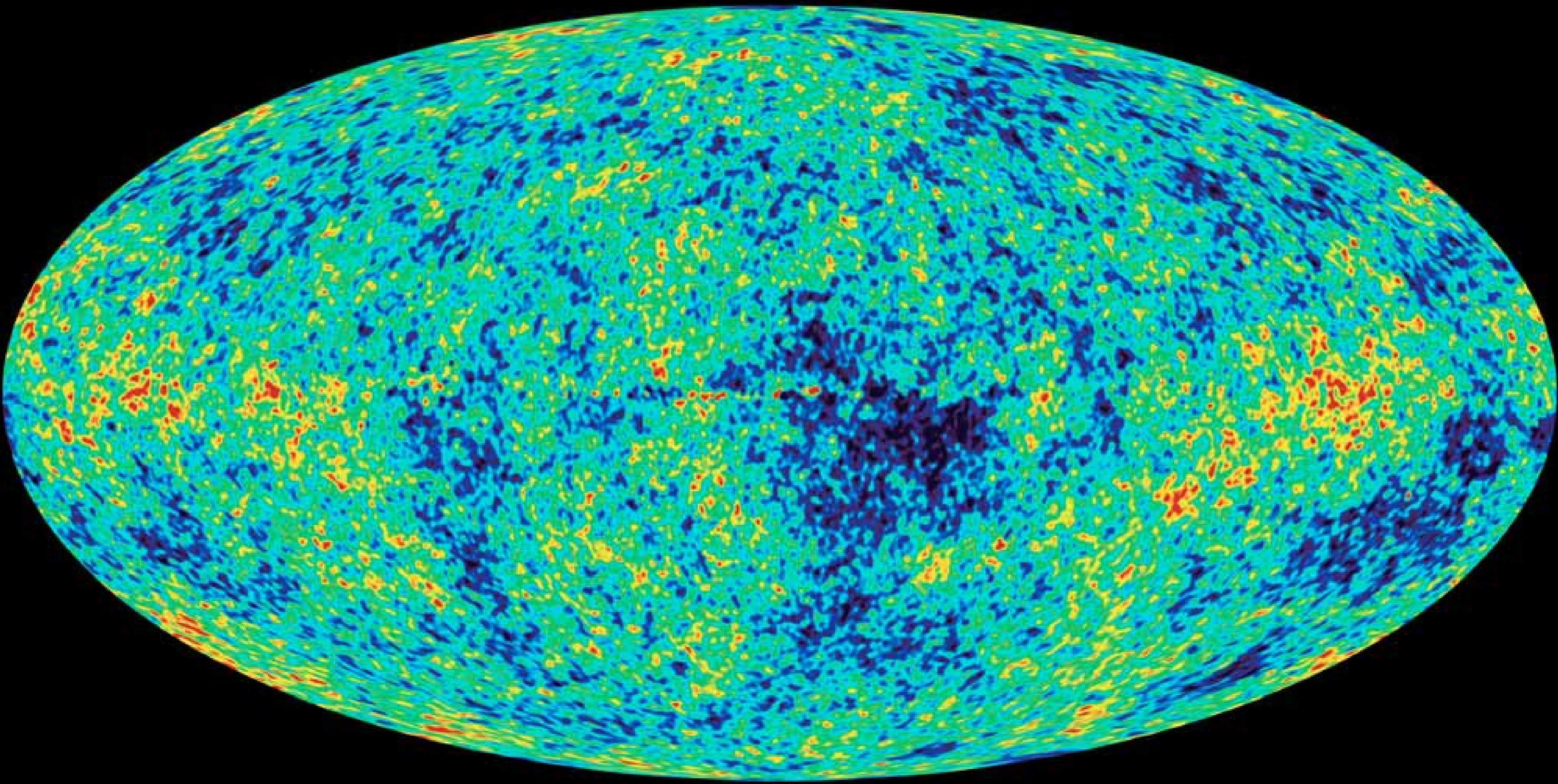
If the fluctuation is on $\sim 10^{-5}$ level, Taylor's series theorem guarantees the non-linear terms are negligible $\sim 10^{-10}$.

Still, considering that the basic equations are fully nonlinear, it is matter of whether we can ignore (or tolerate) the level of nonlinearities.

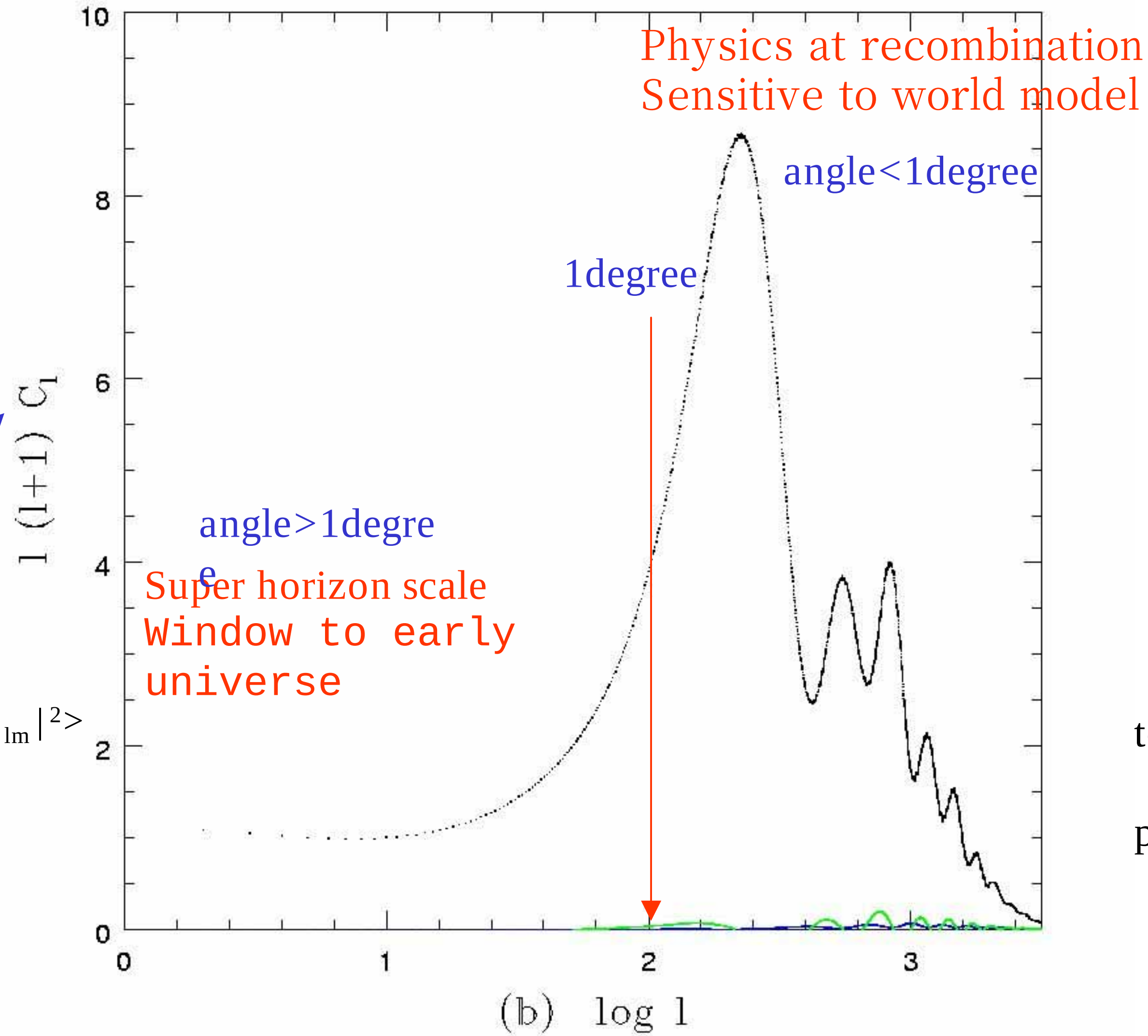
It looks we may currently **assume** linearity in the early universe and in the large-scale in the present era.

If the situation is linear, then we can handle both physics and mathematics very reliably.

WMAP



Theoretical predictions of temperature and polarization anisotropies



$C_l \equiv (2l+1)^{-1} \sum_m <|a_{lm}|^2>$

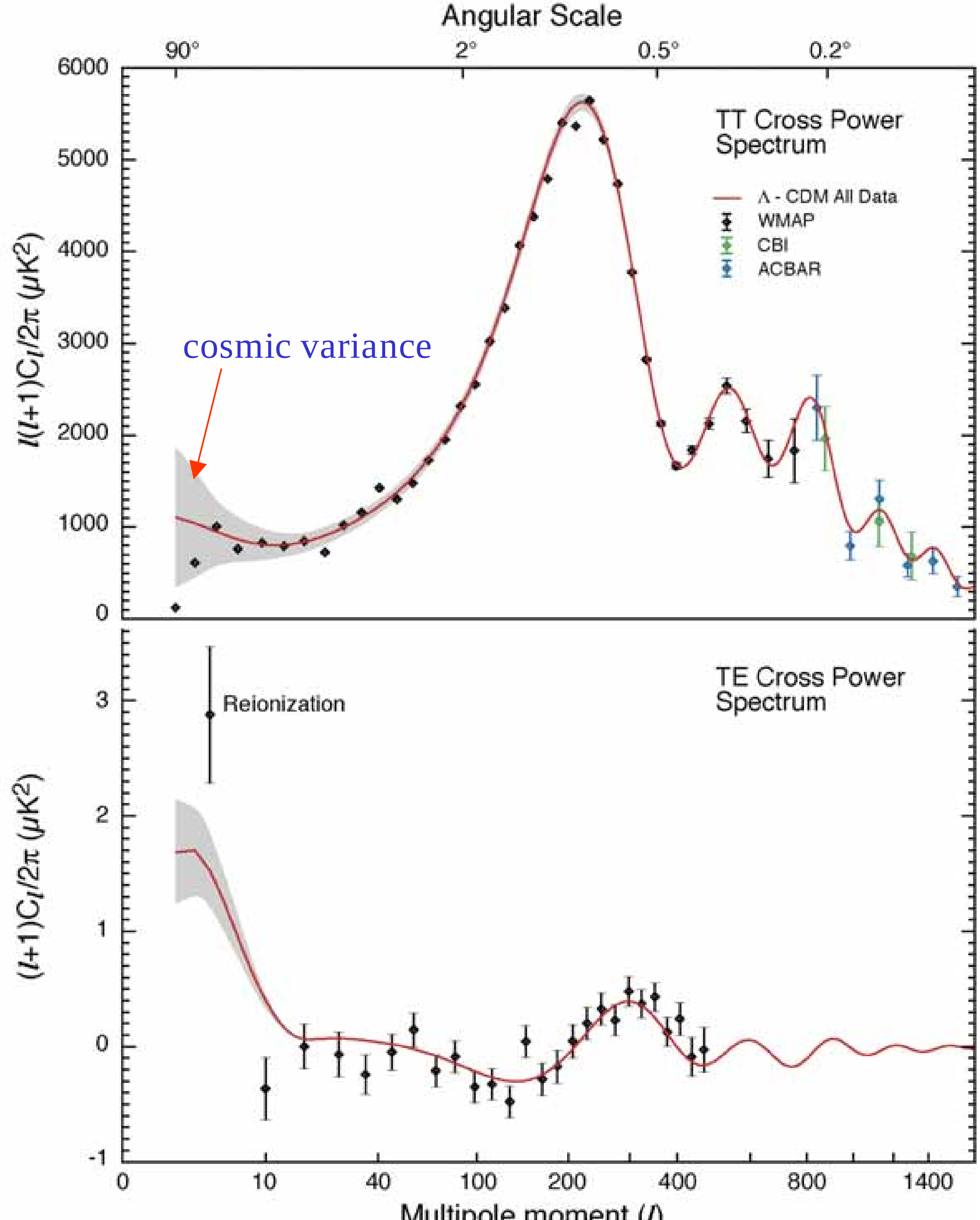
$\delta T / T \equiv \sum a_{lm} Y_{lm}$

temperature ------

polarization ------

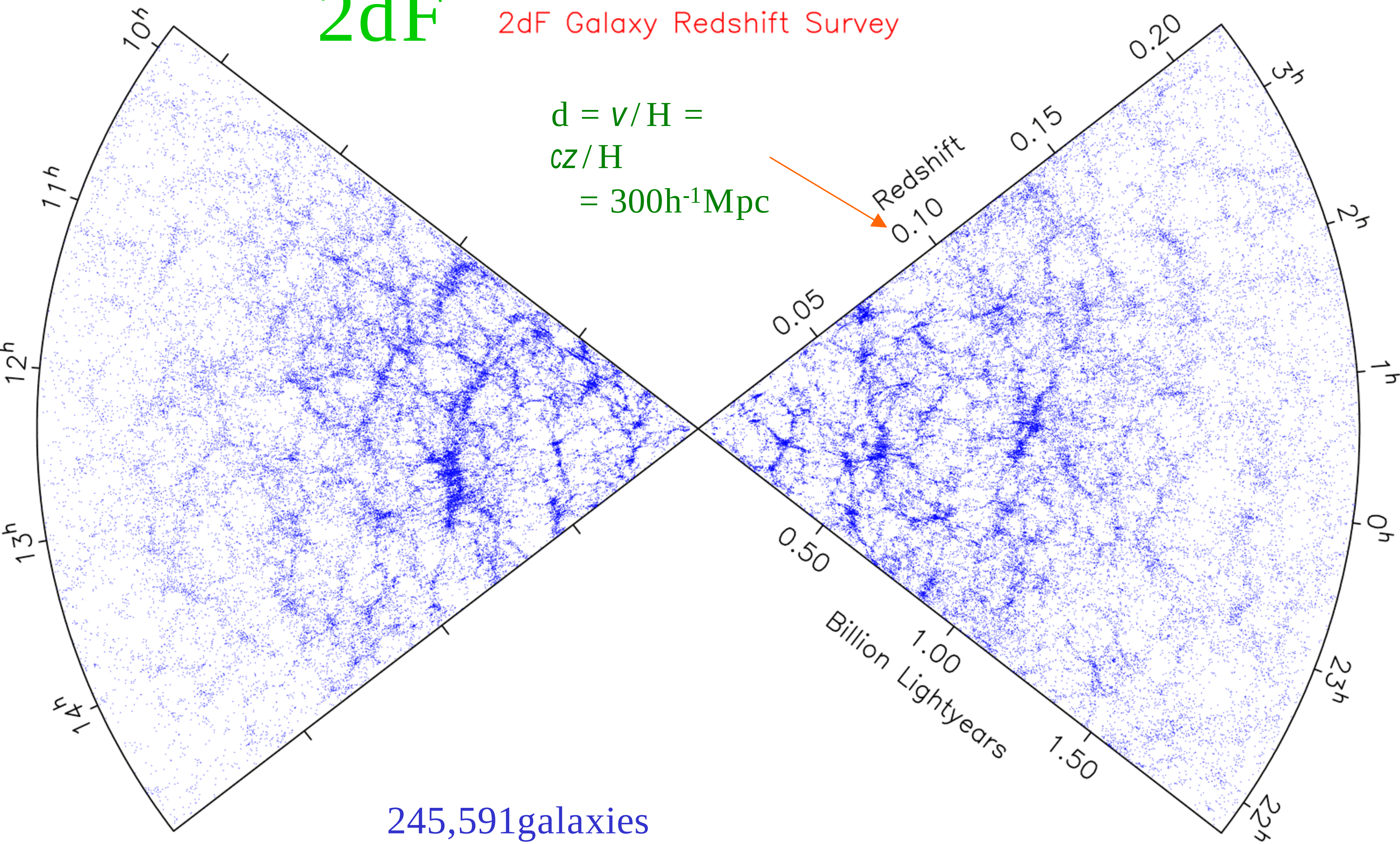
WMAP

Temperature-polarization anisotropies



2dF

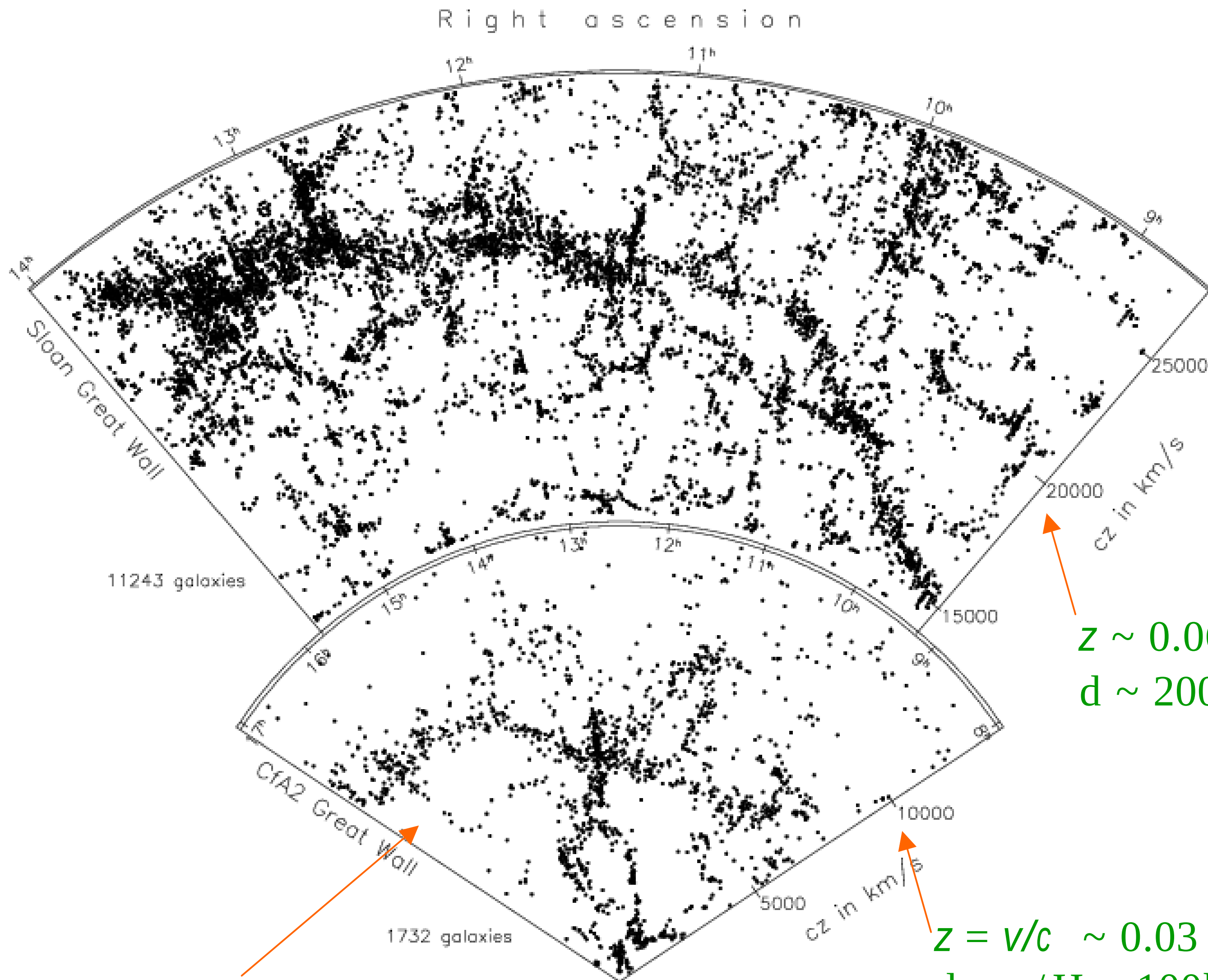
$d = v/H = cz/H = 300h^{-1}\text{Mpc}$



245,591galaxies

Limiting magnitude : 19.45

SDSS



$z \sim 0.06$

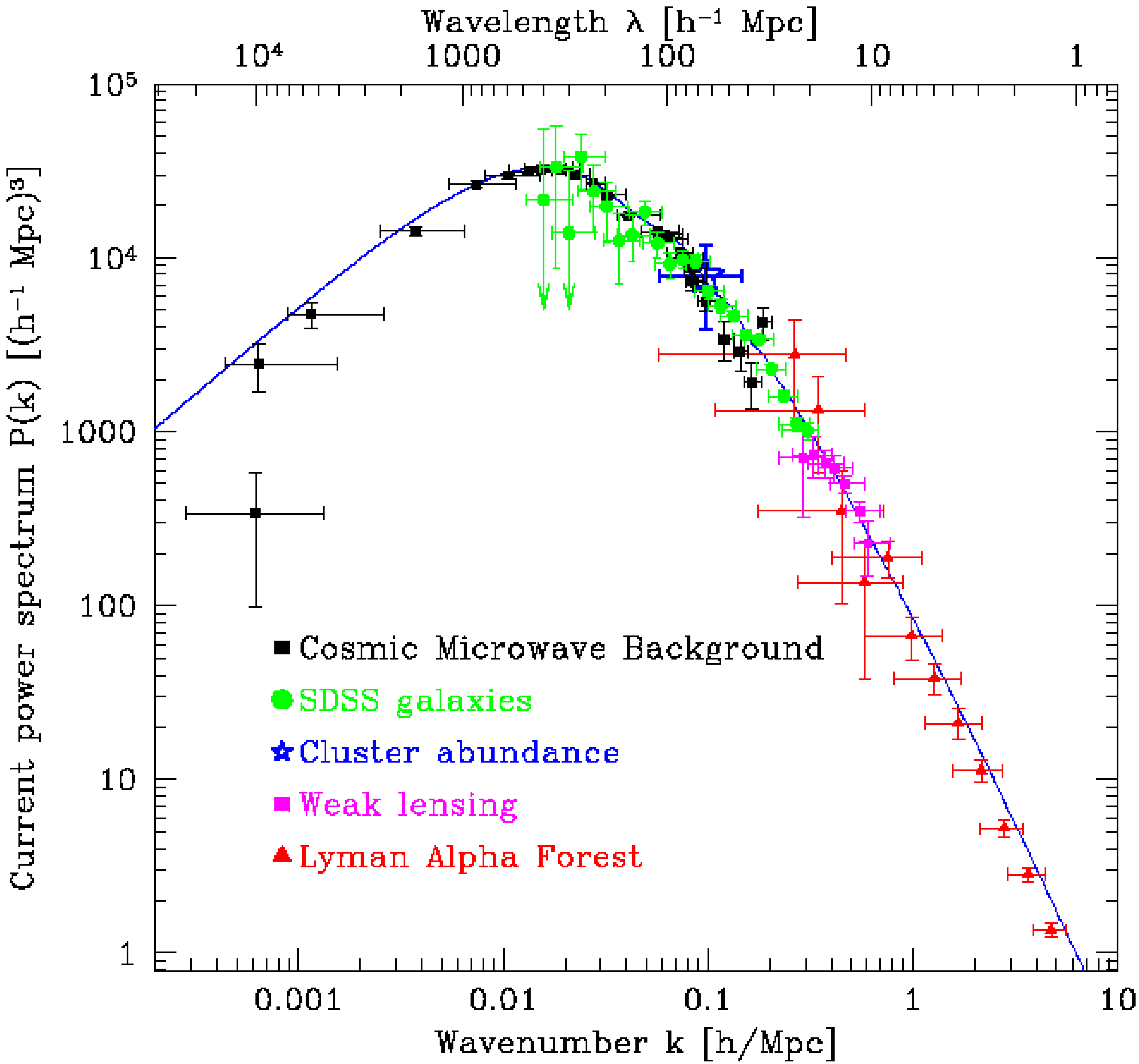
$d \sim 200h^{-1}$Mpc

$z = v/c \sim 0.03$

$d = v/H \sim 100h^{-1}$Mpc

de Lapparent *etal*. (1986)

Density power spectrum



Blue line: 5%atom, 25%DM, 70% DE

Tegmark et al (2002)

Connection and curvature:

Metric:

$$\tilde{g}_{00} = -a^2\left(1+2A\right), \quad \tilde{g}_{0\alpha} = -a^2 B_\alpha, \quad \tilde{g}_{\alpha\beta} = a^2\left(g^{(3)}_{\alpha\beta} + 2C_{\alpha\beta}\right). \tag{75}$$

Inverse metric:

$$\tilde{g}^{00} = -\frac{1}{a^2}\left(1-2A\right), \quad \tilde{g}^{0\alpha} = -\frac{1}{a^2}B^\alpha, \quad \tilde{g}^{\alpha\beta} = \frac{1}{a^2}\left(g^{(3)\alpha\beta} - 2C^{\alpha\beta}\right). \tag{76}$$

Connections:

$$\begin{aligned}
&\tilde{\Gamma}^0_{00} = \frac{a'}{a} + A', \quad \tilde{\Gamma}^0_{0\alpha} = A_{,\alpha} - \frac{a'}{a}B_\alpha, \quad \tilde{\Gamma}^\alpha_{00} = A^{,\alpha} - B^{\alpha\prime} - \frac{a'}{a}B^\alpha, \\
&\tilde{\Gamma}^0_{\alpha\beta} = \frac{a'}{a}g^{(3)}_{\alpha\beta} - 2\frac{a'}{a}g^{(3)}_{\alpha\beta}A + B_{(\alpha|\beta)} + C'_{\alpha\beta} + 2\frac{a'}{a}C_{\alpha\beta}, \\
&\tilde{\Gamma}^\alpha_{0\beta} = \frac{a'}{a}\delta^\alpha_\beta + \frac{1}{2}\left(B_\beta{}^{|\alpha} - B^\alpha{}_{|\beta}\right) + C^{\alpha\prime}_\beta, \quad \tilde{\Gamma}^\alpha_{\beta\gamma} = \Gamma^{(3)\alpha}{}_{\beta\gamma} + \frac{a'}{a}g^{(3)}_{\beta\gamma}B^\alpha + 2C^\alpha_{(\beta|\gamma)} - C_{\beta\gamma}{}^{|\alpha}.
\end{aligned} \tag{77}$$

Curvatures:

$$\begin{aligned}
&\tilde{R}^a{}_{b00} = 0, \quad \tilde{R}^0{}_{00\alpha} = -\left(\frac{a'}{a}\right)' B_\alpha, \quad \tilde{R}^0{}_{0\alpha\beta} = 0, \\
&\tilde{R}^0{}_{\alpha0\beta} = \left(\frac{a'}{a}\right)' g^{(3)}_{\alpha\beta} - \left[\frac{a'}{a}A' + 2\left(\frac{a'}{a}\right)' A\right] g^{(3)}_{\alpha\beta} - A_{,\alpha|\beta} + B'_{(\alpha|\beta)} + \frac{a'}{a}B_{(\alpha|\beta)} + C''_{\alpha\beta} + \frac{a'}{a}C'_{\alpha\beta} + 2\left(\frac{a'}{a}\right)' C_{\alpha\beta}, \\
&\tilde{R}^0{}_{\alpha\beta\gamma} = 2\frac{a'}{a}g^{(3)}_{\alpha[\beta}A_{,\gamma]} - B_{\alpha|[\beta\gamma]} + \frac{1}{2}(B_{\gamma|\alpha\beta} - B_{\beta|\alpha\gamma}) - 2C'_{\alpha[\beta|\gamma]}, \\
&\tilde{R}^\alpha{}_{00\beta} = \left(\frac{a'}{a}\right)' \delta^\alpha_\beta - \frac{a'}{a}A'\delta^\alpha_\beta - A^{|\alpha}{}_\beta + \frac{1}{2}\left(B_\beta{}^{|\alpha} + B^\alpha{}_{|\beta}\right)' + \frac{1}{2}\frac{a'}{a}\left(B_\beta{}^{|\alpha} + B^\alpha{}_{|\beta}\right) + C^{\alpha\prime\prime}_\beta + \frac{a'}{a}C^{\alpha\prime}_\beta, \\
&\tilde{R}^\alpha{}_{0\beta\gamma} = 2\frac{a'}{a}\delta^\alpha_{[\beta}A_{,\gamma]} - B_{[\beta}{}^{|\alpha}{}_{\gamma]} + B^\alpha{}_{|[\beta\gamma]} - 2\left(\frac{a'}{a}\right)^2 \delta^\alpha_{[\beta}B_{\gamma]} - 2C^{\alpha\prime}_{[\beta|\gamma]} \\
&\tilde{R}^\alpha{}_{\beta0\gamma} = \frac{a'}{a}\left(g^{(3)}_{\beta\gamma}A^{,\alpha} - \delta^\alpha_\gamma A_{,\beta}\right) + \left(\frac{a'}{a}\right)' g^{(3)}_{\beta\gamma}B^\alpha - \left(\frac{a'}{a}\right)^2\left(g^{(3)}_{\beta\gamma}B^\alpha - \delta^\alpha_\gamma B_\beta\right) - \frac{1}{2}\left(B_\beta{}^{|\alpha} - B^\alpha{}_{|\beta}\right)_{|\gamma} + C^{\alpha\prime}_{\gamma|\beta} - C'_{\beta\gamma}{}^{|\alpha},
\end{aligned}$$

$$\tilde{R}^{\alpha}{}_{\beta\gamma\delta} = R^{(3)\alpha}{}_{\beta\gamma\delta} + \left(\frac{a'}{a}\right)^2 \left(\delta^{\alpha}_{\gamma} g^{(3)}_{\beta\delta} - \delta^{\alpha}_{\delta} g^{(3)}_{\beta\gamma}\right)(1-2A)$$
$$+\frac{1}{2}\frac{a'}{a}\left[g^{(3)}_{\beta\delta}\left(B_{\gamma}{}^{|\alpha} + B^{\alpha}{}_{|\gamma}\right) - g^{(3)}_{\beta\gamma}\left(B_{\delta}{}^{|\alpha} + B^{\alpha}{}_{|\delta}\right) + 2\delta^{\alpha}_{\gamma} B_{(\beta|\delta)} - 2\delta^{\alpha}_{\delta} B_{(\beta|\gamma)}\right]$$
$$+\frac{a'}{a}\left[g^{(3)}_{\beta\delta} C^{\alpha\prime}_{\gamma} - g^{(3)}_{\beta\gamma} C^{\alpha\prime}_{\delta} + \delta^{\alpha}_{\gamma} C'_{\beta\delta} - \delta^{\alpha}_{\delta} C'_{\beta\gamma} + 2\frac{a'}{a}\left(\delta^{\alpha}_{\gamma} C_{\beta\delta} - \delta^{\alpha}_{\delta} C_{\beta\gamma}\right)\right]$$
$$+2C^{\alpha}_{(\beta|\delta)\gamma} - 2C^{\alpha}_{(\beta|\gamma)\delta} + C_{\beta\gamma}{}^{|\alpha}{}_{\delta} - C_{\beta\delta}{}^{|\alpha}{}_{\gamma}, \tag{78}$$

$$\tilde{R}_{00} = -3\left(\frac{a'}{a}\right)' + 3\frac{a'}{a}A' + \Delta A - B^{\alpha\prime}{}_{|\alpha} - \frac{a'}{a}B^{\alpha}{}_{|\alpha} - C^{\alpha\prime\prime}_{\alpha} - \frac{a'}{a}C^{\alpha\prime}_{\alpha},$$
$$\tilde{R}_{0\alpha} = 2\frac{a'}{a}A_{,\alpha} - \left(\frac{a'}{a}\right)' B_{\alpha} - 2\left(\frac{a'}{a}\right)^2 B_{\alpha} + \frac{1}{2}\Delta B_{\alpha} - \frac{1}{2}B^{\beta}{}_{|\alpha\beta} - C^{\beta\prime}_{\beta|\alpha} + C'_{\alpha\beta}{}^{|\beta},$$
$$\tilde{R}_{\alpha\beta} = 2Kg^{(3)}_{\alpha\beta} + \left[\left(\frac{a'}{a}\right)' + 2\left(\frac{a'}{a}\right)^2\right] g^{(3)}_{\alpha\beta}(1-2A) - \frac{a'}{a}A' g^{(3)}_{\alpha\beta} - A_{,\alpha|\beta} + B'_{(\alpha|\beta)} + 2\frac{a'}{a}B_{(\alpha|\beta)} + \frac{a'}{a}g^{(3)}_{\alpha\beta}B^{\gamma}{}_{|\gamma}$$
$$+C''_{\alpha\beta} + 2\frac{a'}{a}C'_{\alpha\beta} + 2\left[\left(\frac{a'}{a}\right)' + 2\left(\frac{a'}{a}\right)^2\right]C_{\alpha\beta} + \frac{a'}{a}g^{(3)}_{\alpha\beta}C^{\gamma\prime}_{\gamma} + 2C^{\gamma}_{(\alpha|\beta)\gamma} - C^{\gamma}_{\gamma|\alpha\beta} - \Delta C_{\alpha\beta}, \tag{79}$$
$$\tilde{R} = \frac{1}{a^2}\Bigg\{6\left[\left(\frac{a'}{a}\right)' + \left(\frac{a'}{a}\right)^2 + K\right] - 6\frac{a'}{a}A' - 12\left[\left(\frac{a'}{a}\right)' + \left(\frac{a'}{a}\right)^2\right]A - 2\Delta A$$
$$+2B^{\alpha\prime}{}_{|\alpha} + 6\frac{a'}{a}B^{\alpha}{}_{|\alpha} + 2C^{\alpha\prime\prime}_{\alpha} + 6\frac{a'}{a}C^{\alpha\prime}_{\alpha} - 4KC^{\alpha}_{\alpha} - 2\Delta C^{\alpha}_{\alpha} + 2C^{\alpha\beta}{}_{|\alpha\beta}\Bigg\}. \tag{80}$$

It is convenient to have:

$$B^{\alpha}{}_{|\beta\gamma} = B^{\alpha}{}_{|\gamma\beta} - R^{(3)\alpha}{}_{\delta\beta\gamma}B^{\delta}, \quad B_{\alpha|\beta\gamma} = B_{\alpha|\gamma\beta} + R^{(3)\delta}{}_{\alpha\beta\gamma}B_{\delta},$$
$$R^{(3)\alpha}{}_{\beta\gamma\delta} = \frac{1}{6}R^{(3)}\left(\delta^{\alpha}_{\gamma}g^{(3)}_{\beta\delta} - \delta^{\alpha}_{\delta}g^{(3)}_{\beta\gamma}\right), \quad R^{(3)}_{\alpha\beta} = \frac{1}{3}R^{(3)}g^{(3)}_{\alpha\beta}, \quad R^{(3)} = 6K. \tag{81}$$

Time derivative convention:

$$\dot{A} \equiv \frac{\partial A}{\partial t}, \quad A' \equiv \frac{\partial A}{\partial \eta}, \quad cdt \equiv ad\eta. \tag{82}$$

In decomposed form Ricci and scalar curvatures are:

$$R^0_0 = \frac{1}{a^2}\left[3\left(\frac{a'}{a}\right)' - 6\left(\frac{a'}{a}\right)'\alpha + 3\varphi'' + 3\frac{a'}{a}(\varphi' - \alpha') - \Delta\alpha + \Delta(\beta + \gamma')' + \frac{a'}{a}\Delta(\beta + \gamma')\right],$$

$$R^0_\alpha = \frac{1}{a^2}\left\{2\left[\varphi' - \frac{a'}{a}\alpha - K(\beta + \gamma')\right]_{,\alpha} - \frac{1}{2}(\Delta + 2K)\left(B^{(v)}_\alpha + C^{(v)\prime}_\alpha\right)\right\},$$

$$\begin{aligned} R^\alpha_\beta = \frac{1}{a^2}\Bigg\{\Bigg[& \left[\left(\frac{a'}{a}\right)' + 2\left(\frac{a'}{a}\right)^2 + 2K\right]\delta^\alpha_\beta \\ & + \left\{\varphi'' + \frac{a'}{a}\left[5\varphi' - \alpha' + \Delta(\beta + \gamma')\right] - \Delta\varphi - 2\left[\left(\frac{a'}{a}\right)' + 2\left(\frac{a'}{a}\right)^2\right]\alpha - 4K\varphi\right\}\delta^\alpha_\beta \\ & + \left[(\beta + \gamma')' + 2\frac{a'}{a}(\beta + \gamma') - \alpha - \varphi\right]^{|\alpha}_{\;\;\beta} + \frac{1}{2a^2}\left\{a^2\left[B^{(v)\alpha}{}_{|\beta} + B^{(v)|\alpha}_\beta + \left(C^{(v)\alpha}{}_{|\beta} + C^{(v)|\alpha}_\beta\right)'\right]\right\}' \\ & + C^{(t)\alpha\prime\prime}{}_\beta + 2\frac{a'}{a}C^{(t)\alpha\prime}{}_\beta - (\Delta - 2K)C^{(t)\alpha}{}_\beta\Bigg]\Bigg\}, \end{aligned} \tag{83}$$

$$\begin{aligned} R = \frac{1}{a^2}\Bigg\{ & 6\left[\left(\frac{a'}{a}\right)' + \left(\frac{a'}{a}\right)^2 + K\right] + 6\varphi'' + 6\frac{a'}{a}(3\varphi' - \alpha') \\ & -12\left[\left(\frac{a'}{a}\right)' + \left(\frac{a'}{a}\right)^2\right]\alpha - 12K\varphi + 2\Delta\left[(\beta + \gamma')' + 3\frac{a'}{a}(\beta + \gamma') - \alpha - 2\varphi\right]\Bigg\}. \end{aligned} \tag{84}$$

Kinematic quantities:

Metric quantities:

$$N = a(1+\alpha) = \text{lapse function},$$
$$N_\alpha = -a^2\left(\beta_{,\alpha} + B^{(v)}_\alpha\right) = \text{shift vector},$$
$$h_{\alpha\beta} = a^2\left[g^{(3)}_{\alpha\beta} + 2\left(\varphi g^{(3)}_{\alpha\beta} + \gamma_{,\alpha|\beta} + C^{(v)}_{(\alpha|\beta)} + C^{(t)}_{\alpha\beta}\right)\right] = \text{three space metric}. \quad (85)$$

Kinematic quantites in the normal frame:

$$\tilde{\theta} = -K = 3H - \kappa,$$
$$\tilde{\sigma}_{\alpha\beta} = -\bar{K}_{\alpha\beta} = \chi_{,\alpha|\beta} - \frac{1}{3}g^{(3)}_{\alpha\beta}\Delta\chi + a\Psi^{(v)}_{(\alpha|\beta)} + a^2\dot{C}^{(t)}_{\alpha\beta},$$
$$\tilde{\omega}_{\alpha\beta} = 0,$$
$$\tilde{a}_\alpha = (\ln N)_{,\alpha} = \alpha_{,\alpha},$$
$$R^{(h)} = \frac{1}{a^2}\left[6\bar{K} - 4\left(\Delta + 3\bar{K}\right)\varphi\right], \quad (86)$$

where we introduced

$$\chi \equiv a(\beta + a\dot{\gamma}), \quad \Psi^{(v)}_\alpha \equiv B^{(v)}_\alpha + a\dot{C}^{(v)}_\alpha, \quad \kappa \equiv \delta K = 3H\alpha - 3\dot{\varphi} - \frac{\Delta}{a^2}\chi. \quad (87)$$

Thus

$$\chi = \text{shear}$$
$$\kappa = \text{perturbed expansion}$$
$$\varphi = \text{perturbed curvature}$$

Energy-momentum tensor:

Perturbations:

$$\tilde{T}_{ab}(\mathbf{x},t) = T_{ab}(t) + \delta T_{ab}(\mathbf{x},t). \tag{88}$$

Fluid quantities:

$$\tilde{T}^0_0 \equiv -\mu - \delta\mu, \quad \tilde{T}^0_\alpha \equiv Q_\alpha, \quad \tilde{T}^\alpha_\beta \equiv (p + \delta p)\,\delta^\alpha_\beta + \Pi^\alpha_\beta. \tag{89}$$

Decomposition:

$$Q_\alpha \equiv (\mu + p)\left(-v_{,\alpha} + v^{(v)}_\alpha\right),$$
$$\Pi_{\alpha\beta} \equiv \frac{1}{a^2}\left(\Pi_{,\alpha|\beta} - \frac{1}{3}g^{(3)}_{\alpha\beta}\Delta\Pi\right) + \frac{1}{a}\Pi^{(v)}_{(\alpha|\beta)} + \Pi^{(t)}_{\alpha\beta}. \tag{90}$$

These variable are frame independent.

The frame vector $\tilde{u}_a$:

$$\tilde{u}_0 \equiv -a\,(1 + A)\,, \quad \tilde{u}_\alpha \equiv a\,(V_\alpha - B_\alpha)\,; \quad \tilde{u}^0 = \frac{1}{a}(1 - A)\,, \quad \tilde{u}^\alpha = \frac{1}{a}V^\alpha, \tag{91}$$

The normal-frame vector $\tilde{n}_a$ sets $\tilde{n}_\alpha \equiv 0$, thus $V_\alpha = B_\alpha$:

$$\tilde{n}_0 \equiv -a\,(1 + A)\,, \quad \tilde{n}_\alpha \equiv 0; \quad \tilde{n}^0 = \frac{1}{a}(1 - A)\,, \quad \tilde{n}^\alpha = \frac{1}{a}B^\alpha. \tag{92}$$

The energy-frame vector sets $\tilde{q}_a \equiv 0$. (9,89) gives:

$$V_\alpha - B_\alpha \equiv \frac{1}{\mu + p}Q_\alpha. \tag{93}$$

2. Basic Equations:

Background:

$\tilde{G}^0_0$ and $\tilde{G}^\alpha_\alpha - 2\tilde{G}^0_0$:

$$\boxed{\begin{aligned} H^2 &= \frac{8\pi G}{3}\mu - \frac{K}{a^2} + \frac{\Lambda}{3}, \qquad (94) \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}\left(\mu + 3p\right) + \frac{\Lambda}{3}. \qquad (95) \end{aligned}}$$

$\tilde{T}^b_{0;b} = 0$:

$$\boxed{\dot{\mu} + 3H\left(\mu + p\right) = 0.} \qquad (96)$$

(96) follows from (94,95).

Scalar-type perturbation: (Bardeen 1988) [5, 13]

Definition of κ:

$$\kappa \equiv 3H\alpha - 3\dot{\varphi} - \frac{\Delta}{a^2}\chi. \tag{97}$$

$\tilde{G}^0_0$, $\tilde{G}^0_\alpha$, $\tilde{G}^\alpha_\beta - \frac{1}{3}\delta^\alpha_\beta \tilde{G}^\gamma_\gamma$ and $\tilde{G}^\alpha_\alpha - \tilde{G}^0_0$:

$$H\kappa + \frac{\Delta + 3K}{a^2}\varphi = -4\pi G\delta\mu, \tag{98}$$

$$\kappa + \frac{\Delta + 3K}{a^2}\chi = 12\pi G(\mu + p)av, \tag{99}$$

$$\dot{\kappa} + 2H\kappa + \left(3\dot{H} + \frac{\Delta}{a^2}\right)\alpha = 4\pi G\left(\delta\mu + 3\delta p\right), \tag{100}$$

$$\dot{\chi} + H\chi - \varphi - \alpha = 8\pi G\Pi. \tag{101}$$

$\tilde{T}^b_{0;b} = 0$ and $\tilde{T}^b_{\alpha;b} = 0$:

$$\delta\dot{\mu} + 3H\left(\delta\mu + \delta p\right) = (\mu + p)\left(\kappa - 3H\alpha + \frac{1}{a}\Delta v\right), \tag{102}$$

$$\frac{[a^4(\mu + p)v]^{\cdot}}{a^4(\mu + p)} = \frac{1}{a}\alpha + \frac{1}{a(\mu + p)}\left(\delta p + \frac{2}{3}\frac{\Delta + 3K}{a^2}\Pi\right). \tag{103}$$

(97-103) provide a complete set for scalar-type perturbation.

Vector-type perturbation:

$\tilde{G}^0_\alpha$, $\tilde{G}^\alpha_\beta$ and $\tilde{T}^b_{\alpha;b} = 0$:

$$\frac{\Delta + 2K}{2a^2}\Psi^{(v)}_\alpha = -8\pi G(\mu + p)v^{(v)}_\alpha, \quad (104)$$

$$\dot{\Psi}^{(v)}_\alpha + 2H\Psi^{(v)}_\alpha = 8\pi G\Pi^{(v)}_\alpha, \quad (105)$$

$$\frac{[a^4(\mu + p)v^{(v)}_\alpha]^\cdot}{a^4(\mu + p)} = -\frac{\Delta + 2K}{2a^2}\frac{\Pi^{(v)}_\alpha}{\mu + p}. \quad (106)$$

For vanishing anisotropic stress:

Angular momentum $\sim \left[a^3(\mu + p) \cdot a \cdot v^{(v)}_\alpha\right] \sim$ conserved.

Tensor-type perturbation:

$\tilde{G}^\alpha_\beta$:

$$\ddot{C}^{(t)}_{\alpha\beta} + 3H\dot{C}^{(t)}_{\alpha\beta} - \frac{\Delta - 2K}{a^2}C^{(t)}_{\alpha\beta} = 8\pi G\Pi^{(t)}_{\alpha\beta}. \quad (107)$$

For $K = 0$:

$$\frac{1}{a^3}\left(a^3\dot{C}^{(t)}_{\alpha\beta}\right)^\cdot - \frac{\Delta}{a^2}C^{(t)}_{\alpha\beta} = \text{stress}. \quad (108)$$

$C^{(t)}_{\alpha\beta}$ is conserved in the super-horizon scale.

Derivation of (101,105,107):

$\tilde{G}^{\alpha}_{\beta} - \frac{1}{3}\delta^{\alpha}_{\beta}\tilde{G}^{\gamma}_{\gamma}$ or (38) gives:

$$\frac{1}{a^2}\left(\nabla_\alpha\nabla_\beta - \frac{1}{3}g^{(3)}_{\alpha\beta}\Delta\right)(\dot{\chi} + H\chi - \varphi - \alpha - 8\pi G\Pi)$$
$$+\frac{1}{a^3}\left(a^2\Psi^{(v)}_{(\alpha|\beta)}\right)^{\cdot} - 8\pi G\frac{1}{a}\Pi^{(v)}_{(\alpha|\beta)}$$
$$+\ddot{C}^{(t)}_{\alpha\beta} + 3H\dot{C}^{(t)}_{\alpha\beta} - \frac{\Delta - 2K}{a^2}C^{(t)}_{\alpha\beta} - 8\pi G\Pi^{(t)}_{\alpha\beta} = 0. \quad (109)$$

We can decompose (109) to three different types of perturbations:
First, by applying ∇^α on (109) we can derive an equation.
Second, by applying $\nabla^\alpha\nabla^\beta$ on (109) we can derive another equation.
From these three equations we can show that the three perturbation types decouple from each other and give (101,105,107).

Scalar field:

Action:

$$\tilde{S} = \int \left[\frac{1}{16\pi G} \tilde{R} - \frac{1}{2} \tilde{\phi}^{;a} \tilde{\phi}_{,a} - V(\tilde{\phi}) \right] \sqrt{-\tilde{g}} d^4 x. \tag{110}$$

Energy-momentum tensor:

$$\tilde{T}_{ab} = \tilde{\phi}_{,a} \tilde{\phi}_{,b} - \left(\frac{1}{2} \tilde{\phi}^{;c} \tilde{\phi}_{,c} + \tilde{V} \right) \tilde{g}_{ab}. \tag{111}$$

Equation of motion: $(\tilde{V}_{,\tilde{\phi}} \equiv \frac{\partial \tilde{V}}{\partial \tilde{\phi}})$

$$\tilde{\phi}^{;c}{}_{c} = \tilde{V}_{,\tilde{\phi}}. \tag{112}$$

Perturbation:

$$\tilde{\phi}(\mathbf{x},t) = \phi(t) + \delta\phi(\mathbf{x},t). \tag{113}$$

Equation of motion:

Background:

$$\boxed{\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0.} \tag{114}$$

Perturbation:

$$\boxed{\delta\ddot{\phi} + 3H\delta\dot{\phi} - \frac{\Delta}{a^2}\delta\phi + V_{,\phi\phi}\delta\phi = \dot{\phi}\left(\kappa + \dot{\alpha}\right) + \left(2\ddot{\phi} + 3H\dot{\phi}\right)\alpha.} \tag{115}$$

Fluid quantities:

$$\boxed{\mu = \frac{1}{2}\dot{\phi}^2 + V, \quad p = \frac{1}{2}\dot{\phi}^2 - V,}$$

$$\boxed{\begin{gathered}\delta\mu = \dot{\phi}\delta\dot{\phi} - \dot{\phi}^2\alpha + V_{,\phi}\delta\phi, \quad \delta p = \dot{\phi}\delta\dot{\phi} - \dot{\phi}^2\alpha - V_{,\phi}\delta\phi, \quad (\mu + p)v = \frac{1}{a}\dot{\phi}\delta\phi, \\ v_\alpha^{(v)} = 0, \quad \Pi_{\alpha\beta} = 0.\end{gathered}} \tag{116}$$

Notice that

$$\delta\phi = 0 \quad \text{implies} \quad v = 0. \tag{117}$$

Derivation of (115,116):

(112) gives:

$$\begin{aligned}\tilde{\phi}^{;c}{}_{c} &= \tilde{g}^{cd}\tilde{\phi}_{,c;d} = \tilde{g}^{cd}\left(\tilde{\phi}_{,cd} - \tilde{\Gamma}^{e}_{cd}\tilde{\phi}_{,e}\right) \\ &= \tilde{g}^{00}\left(\tilde{\phi}_{,00} - \tilde{\Gamma}^{e}_{00}\tilde{\phi}_{,e}\right) + \tilde{g}^{\alpha\beta}\left(\tilde{\phi}_{,\alpha\beta} - \tilde{\Gamma}^{e}_{\alpha\beta}\tilde{\phi}_{,e}\right) \\ &= \tilde{g}^{00}\left(\tilde{\phi}_{,00} - \tilde{\Gamma}^{0}_{00}\tilde{\phi}_{,0}\right) + \tilde{g}^{\alpha\beta}\left(\tilde{\phi}_{,\alpha\beta} - \tilde{\Gamma}^{0}_{\alpha\beta}\tilde{\phi}_{,0} - \tilde{\Gamma}^{\gamma}_{\alpha\beta}\tilde{\phi}_{,\gamma}\right) \\ &= \tilde{V}_{,\tilde{\phi}} = \tilde{V}_{,\tilde{\phi}}(\tilde{\phi}) = \tilde{V}_{,\tilde{\phi}}(\phi) + (\tilde{V}_{,\tilde{\phi}})_{,\phi}\delta\phi = V_{,\phi}(\phi) + V_{,\phi\phi}\delta\phi. \end{aligned} \tag{118}$$

Using (76,77) we can derive (114,115)

From (89,111) we have:

$$\begin{aligned}\tilde{T}^{0}_{0} &= -\mu - \delta\mu = \tilde{g}^{0c}\tilde{\phi}_{,c}\tilde{\phi}_{,0} - \left[\frac{1}{2}\tilde{g}^{cd}\tilde{\phi}_{,c}\tilde{\phi}_{,d} + V(\tilde{\phi})\right] = \frac{1}{2}\tilde{g}^{00}\tilde{\phi}_{,0}\tilde{\phi}_{,0} - V(\tilde{\phi}) \\ &= -\frac{1}{2}\frac{1}{a^2}\left(1-2\alpha\right)\left(\phi+\delta\phi\right)_{,0}\left(\phi+\delta\phi\right)_{,0} - V(\phi) - V_{,\phi}\delta\phi \\ &= -\frac{1}{2}\dot{\phi}^2 - V(\phi) - \dot{\phi}\delta\dot{\phi} + \alpha\dot{\phi}^2 - V_{,\phi}\delta\phi. \end{aligned} \tag{119}$$

Thus we have μ and $\delta\mu$ in (116).

Multi-component case:

Energy-momentum tensor in (41) gives:

$$\mu = \sum_k \mu_{(k)}, \quad p = \sum_k p_{(k)}; \tag{120}$$

$$\delta\mu = \sum_k \delta\mu_{(k)}, \quad \delta p = \sum_k \delta p_{(k)}, \quad (\mu + p)v = \sum_k (\mu_{(k)} + p_{(k)})v_{(k)},$$

$$(\mu + p)v^{(v)} = \sum_k (\mu_{(k)} + p_{(k)})v^{(v)}_{(k)}, \quad \Pi_{\alpha\beta} = \sum_k \Pi_{(k)\alpha\beta};$$

$$\tilde{I}_{(i)a} \equiv I_{(i)a} + \delta I_{(i)a}, \quad \delta I_{(i)\alpha} \equiv \delta I_{(i),\alpha} + \delta I^{(v)}_{(i)\alpha}, \quad \delta I^{(v)\alpha}_{(i)\ \ |\alpha} \equiv 0. \tag{121}$$

Energy and momentum conservations (44,45) give:

$$\dot{\mu}_{(i)} + 3H\left(\mu_{(i)} + p_{(i)}\right) = -\frac{1}{a} I_{(i)0}, \tag{122}$$

$$\delta\dot{\mu}_{(i)} + 3H\left(\delta\mu_{(i)} + \delta p_{(i)}\right) = \left(\mu_{(i)} + p_{(i)}\right)\left(\kappa - 3H\alpha + \frac{1}{a}\Delta v_{(i)}\right) - \frac{1}{a}\delta I_{(i)0}, \tag{123}$$

$$\frac{[a^4(\mu_{(i)} + p_{(i)})v_{(i)}]^{\cdot}}{a^4(\mu_{(i)} + p_{(i)})} = \frac{1}{a}\alpha + \frac{1}{a(\mu_{(i)} + p_{(i)})}\left(\delta p_{(i)} + \frac{2}{3}\frac{\Delta + 3K}{a^2}\Pi_{(i)} - \delta I_{(i)}\right). \tag{124}$$

$$\frac{[a^4(\mu_{(i)} + p_{(i)})v^{(v)}_{(i)\alpha}]^{\cdot}}{a^4(\mu_{(i)} + p_{(i)})} = -\frac{\Delta + 2K}{2a^2}\frac{\Pi^{(v)}_{(i)\alpha}}{\mu_{(i)} + p_{(i)}} + \frac{1}{a}\frac{\delta I^{(v)}_{(i)\alpha}}{\mu_{(i)} + p_{(i)}}. \tag{125}$$

(97-103,123,124) provide a complete set for scalar-type perturbation.

Scalar fields:

Equation of motion:

$$\tilde{\phi}_{(i)}{}^{;c}{}_{c} = \tilde{V}_{,\tilde{\phi}_{(i)}}. \tag{126}$$

Perturb:

$$\tilde{\phi}_{(i)}(\mathbf{x}, t) = \phi_{(i)}(t) + \delta\phi_{(i)}(\mathbf{x}, t). \tag{127}$$

Equations of motion:

$$\ddot{\phi}_{(i)} + 3H\dot{\phi}_{(i)} + V_{,\phi_{(i)}} = 0, \tag{128}$$

$$\delta\ddot{\phi}_{(i)} + 3H\delta\dot{\phi}_{(i)} - \frac{\Delta}{a^2}\delta\phi_{(i)} + \sum_k V_{,\phi_{(i)}\phi_{(k)}}\delta\phi_{(k)} = \dot{\phi}_{(i)}\left(\kappa + \dot{\alpha}\right) + \left(2\ddot{\phi}_{(i)} + 3H\dot{\phi}_{(i)}\right)\alpha. \tag{129}$$

Fluid quantities:

$$\mu = \frac{1}{2}\sum_k \dot{\phi}_{(k)}^2 + V, \quad p = \frac{1}{2}\sum_k \dot{\phi}_{(k)}^2 - V,$$

$$\delta\mu = \sum_k \left(\dot{\phi}_{(k)}\delta\dot{\phi}_{(k)} - \dot{\phi}_{(k)}^2\alpha + V_{,\phi_{(k)}}\delta\phi_{(k)}\right),$$

$$\delta p = \sum_k \left(\dot{\phi}_{(k)}\delta\dot{\phi}_{(k)} - \dot{\phi}_{(k)}^2\alpha - V_{,\phi_{(k)}}\delta\phi_{(k)}\right),$$

$$(\mu + p)v = \frac{1}{a}\sum_k \dot{\phi}_{(k)}\delta\phi_{(k)}, \quad v_\alpha^{(v)} = 0, \quad \Pi_{\alpha\beta} = 0. \tag{130}$$

3. Gauge Issue

- Einstein gravity has spacetime covariance.
- Coordinate invariance → more variables than equations.

  ⇒ Gauge freedom: freedom to choose some conditions.

  "A gauge transformation can be thought of as a coordinate transformation induced by a change in the correspondence between the physical perturbed spacetime and the fictitious background spacetime introduced to define the perturbations."

  J. M. Bardeen (1988)

- Spatial gauge freedom: trivial in Friedmann background
- Temporal gauge freedom: affect scalar-type mode
- Exist several fundamental temporal gauge conditions.

  Except for the synchronous gauge, the other gauge completely removes the gauge mode

  → gauge-invariant!

- Fixing gauge → lose no generality.
- Physics is gauge invariant, *i.e.*, does not depend on the gauge condition we choose.
- A known solution in a gauge → all solutions in every gauge.
- Practically, important to take a gauge which suits the problem.
- Usually, we do not know the suitable condition, *a priori*.

Gauge transformation:

Transformation between two coordinates x^a and $\hat{x}^a$:

$$\hat{x}^a \equiv x^a + \tilde{\xi}^a(x^e). \tag{131}$$

Tensor transformation property between x^a and $\hat{x}^a$ spacetimes.

$$\tilde{\phi}(x^e) = \hat{\tilde{\phi}}(\hat{x}^e), \quad \tilde{v}_a(x^e) = \frac{\partial \hat{x}^b}{\partial x^a}\hat{\tilde{v}}_b(\hat{x}^e), \quad \tilde{t}_{ab}(x^e) = \frac{\partial \hat{x}^c}{\partial x^a}\frac{\partial \hat{x}^d}{\partial x^b}\hat{\tilde{t}}_{cd}(\hat{x}^e). \tag{132}$$

We have, at the same spacetime point:

$$\begin{aligned}
\hat{\tilde{\phi}}(x^e) &= \tilde{\phi}(x^e) - \tilde{\phi}_{,c}\tilde{\xi}^c, \\
\hat{\tilde{v}}_a(x^e) &= \tilde{v}_a(x^e) - \tilde{v}_{a,b}\tilde{\xi}^b - \tilde{v}_b\tilde{\xi}^b{}_{,a}, \\
\hat{\tilde{t}}_{ab}(x^e) &= \tilde{t}_{ab}(x^e) - 2\tilde{t}_{c(a}\tilde{\xi}^c{}_{,b)} - \tilde{t}_{ab,c}\tilde{\xi}^c.
\end{aligned} \tag{133}$$

From the gauge transformation property of $\tilde{g}_{ab}$:

$$\begin{aligned}
\hat{A} &= A - \left(\tilde{\xi}^{0\prime} + \frac{a'}{a}\tilde{\xi}^0\right), \\
\hat{B}_\alpha &= B_\alpha - \tilde{\xi}^0{}_{,\alpha} + \tilde{\xi}'_\alpha, \\
\hat{C}_{\alpha\beta} &= C_{\alpha\beta} - \frac{a'}{a}\tilde{\xi}^0 g^{(3)}_{\alpha\beta} - \frac{1}{2}g^{(3)}_{\alpha\beta,\gamma}\tilde{\xi}^\gamma - g^{(3)}_{\gamma(\alpha}\tilde{\xi}^\gamma{}_{,\beta)}.
\end{aligned} \tag{134}$$

Thus, even the Friedmann background (in x^a coordinate with $A = B_\alpha = C_{\alpha\beta} = 0$) looks perturbed in $\hat{x}^a$ coordinate, and we do not want to confuse such effects from real perturbations.

From the gauge transformation property of $\tilde{T}_{ab}$:

$$\delta\hat{\mu} = \delta\mu - \mu'\tilde{\xi}^0, \quad \delta\hat{p} = \delta p - p'\tilde{\xi}^0, \quad \hat{Q}_\alpha = Q_\alpha + (\mu + p)\tilde{\xi}^0{}_{,\alpha}, \quad \hat{\Pi}_{\alpha\beta} = \Pi_{\alpha\beta}. \tag{135}$$

Decompose:

$$\tilde{\xi}^0 = \frac{1}{a}\xi^t, \quad \tilde{\xi}_\alpha \equiv \frac{1}{a}\xi_{,\alpha} + \xi^{(v)}_\alpha; \quad \xi^{(v)\alpha}{}_{|\alpha} \equiv 0. \tag{136}$$

We have

$$\begin{aligned}
&\hat{\alpha} = \alpha - \dot{\xi}^t, \quad \hat{\beta} = \beta - \frac{1}{a}\xi^t + a\left(\frac{\xi}{a}\right)^{\cdot}, \quad \hat{\gamma} = \gamma - \frac{1}{a}\xi, \quad \hat{\varphi} = \varphi - H\xi^t, \\
&\hat{\chi} = \chi - \xi^t, \quad \hat{\kappa} = \kappa + \left(3\dot{H} + \frac{\Delta}{a^2}\right)\xi^t, \\
&\delta\hat{\mu} = \delta\mu - \dot{\mu}\xi^t, \quad \delta\hat{p} = \delta p - \dot{p}\xi^t, \quad \hat{v} = v - \frac{1}{a}\xi^t, \quad \hat{\Pi} = \Pi, \quad \delta\hat{\phi} = \delta\phi - \dot{\phi}\xi^t, \\
&\hat{B}^{(v)}_\alpha = B^{(v)}_\alpha + a\dot{\xi}^{(v)}_\alpha, \quad \hat{C}^{(v)}_\alpha = C^{(v)}_\alpha - \xi^{(v)}_\alpha, \quad \hat{\Psi}^{(v)}_\alpha = \Psi^{(v)}_\alpha, \quad \hat{v}^{(v)}_\alpha = v^{(v)}_\alpha, \quad \hat{\Pi}^{(v)}_\alpha = \Pi^{(v)}_\alpha, \\
&\hat{C}^{(t)}_{\alpha\beta} = C^{(t)}_{\alpha\beta}, \quad \hat{\Pi}^{(t)}_{\alpha\beta} = \Pi^{(t)}_{\alpha\beta}.
\end{aligned} \tag{137}$$

Scalar-type:	affected by	ξ^t and ξ
Vector-type:	affected by	$\xi^{(v)}_\alpha$
Tensor-type:	gauge-invariant	

Derivation of (133,134,137):

For $\tilde{v}_a$:

$$\tilde{v}_a(x^e) = \frac{\partial \hat{x}^b}{\partial x^a}\hat{\tilde{v}}_b(\hat{x}^e) = \frac{\partial(x^b + \tilde{\xi}^b)}{\partial x^a}\hat{\tilde{v}}_b(x^e + \tilde{\xi}^e) = \left(\delta^b_a + \tilde{\xi}^b{}_{,a}\right)\left[\hat{\tilde{v}}_b(x^e) + \hat{\tilde{v}}_{b,c}\tilde{\xi}^c\right]$$
$$= \hat{\tilde{v}}_a(x^e) + \tilde{\xi}^b{}_{,a}\hat{\tilde{v}}_b + \hat{\tilde{v}}_{a,c}\tilde{\xi}^c = \hat{\tilde{v}}_a(x^e) + \tilde{\xi}^b{}_{,a}\tilde{v}_b + \tilde{v}_{a,c}\tilde{\xi}^c, \quad (138)$$

thus,

$$\hat{\tilde{v}}_a(x^e) = \tilde{v}_a(x^e) - \tilde{v}_{a,b}\tilde{\xi}^b - \tilde{v}_b\tilde{\xi}^b{}_{,a}. \quad (139)$$

For $\tilde{g}_{00}$:

$$\hat{\tilde{g}}_{00}(x^e) = -\hat{a}^2\left(1 + 2\hat{A}\right) = \tilde{g}_{00} - 2\tilde{g}_{c(0}\tilde{\xi}^c{}_{,0)} - \tilde{g}_{00,c}\tilde{\xi}^c$$
$$= -a^2(1 + 2A) - 2\tilde{g}_{00}\tilde{\xi}^0{}_{,0} - \tilde{g}_{00,0}\tilde{\xi}^0. \quad (140)$$

To the background order we have $\hat{a} = a$, and the perturbed order:

$$\hat{A} = A - \tilde{\xi}^0{}_{,0} - \frac{a_{,0}}{a}\tilde{\xi}^0 = A - \tilde{\xi}^{0\prime} - \frac{a'}{a}\tilde{\xi}^0$$
$$= \alpha - \left(\frac{1}{a}\xi^t\right)' - \frac{a'}{a}\frac{1}{a}\xi^t = \alpha - \dot{\xi}^t. \quad (141)$$

Gauge conditions:

Spatial gauge conditions:

> *"Since the background 3-space is homogeneous and isotropic, the perturbations in all physical quantities must in fact be gauge invariant under purely spatial gauge transformations."*
>
> J. M. Bardeen (1988)

We have two natural spatial gauge fixing conditions:

$$B\text{–gauge}: \quad \beta \equiv 0, \quad B_\alpha^{(v)} \equiv 0 \quad \rightarrow \quad \xi(\mathbf{x}, t) \propto a, \quad \xi_\alpha^{(v)}(\mathbf{x}), \tag{142}$$

$$C\text{–gauge}: \quad \gamma \equiv 0, \quad C_\alpha^{(v)} \equiv 0 \quad \rightarrow \quad \xi = 0, \quad \xi_\alpha^{(v)} = 0. \tag{143}$$

The C-gauge conditions ($C_{\alpha\beta} \equiv \varphi g^{(3)}_{\alpha\beta} + C^{(t)}_{\alpha\beta}$) successfully remove the gauge modes.

The B-gauge conditions ($B_\alpha \equiv 0$) fail to fix the spatial and the rotational gauge modes completely, thus, even after imposing the gauge conditions we still have the remaining gauge modes. For β we have considered a situation where the temporal gauge condition already completely removed ξ^t.

To the linear-order, the variables $\chi \equiv a(\beta + a\dot{\gamma})$ and $\Psi_\alpha^{(v)} \equiv B_\alpha^{(v)} + aC_\alpha^{(v)}$ introduced in eq. (87) are natural and unique spatially gauge-invariant combinations. In the C-gauge we have $\chi = a\beta$ and $\Psi_\alpha^{(v)} = B_\alpha^{(v)}$.

Temporal gauge conditions:

Temporal gauge condition fixes ξ^t.
We can impose any one of the following temporal gauge conditions to be valid at any spacetime point:

synchronous gauge:	$\alpha \equiv 0$	$\rightarrow$	$\xi^t(\mathbf{x})$
comoving gauge:	$v \equiv 0$	$\rightarrow$	$\xi^t = 0$
zero-shear gauge:	$\chi \equiv 0$	$\rightarrow$	$\xi^t = 0$
uniform-expansion gauge:	$\kappa \equiv 0$	$\rightarrow$	$\xi^t = 0$
uniform-curvature gauge:	$\varphi \equiv 0$	$\rightarrow$	$\xi^t = 0$
uniform-density gauge:	$\delta\mu \equiv 0$	$\rightarrow$	$\xi^t = 0$
uniform-pressure gauge:	$\delta p \equiv 0$	$\rightarrow$	$\xi^t = 0$
uniform-field gauge:	$\delta\phi \equiv 0$	$\rightarrow$	$\xi^t = 0$

Except for the synchronous gauge condition, each of the other temporal gauge fixing conditions completely removes the temporal gauge mode.

In the multi-component situation:

Additionally we can impose any one of the following as the temporal gauge condition:

$$\delta\mu_{(i)} \equiv 0, \quad \delta p_{(i)} \equiv 0, \quad v_{(i)} \equiv 0, \quad \delta\phi_{(i)} \equiv 0. \qquad (144)$$

Introduce systematic notations for gauge-invariant combinations:

$$\hat{\varphi}_\chi \equiv \hat{\varphi} - H\hat{\chi} = \varphi - H\xi^t - H\left(\chi - \xi^t\right) = \varphi - H\chi \equiv \varphi_\chi. \tag{145}$$

Gauge-invariance means its values is independent of coordinate.

We have:

$$\varphi_\chi \equiv \varphi - H\chi = \varphi|_{\chi\equiv 0}.$$

- φ_χ is *the same* as φ variable in the zero-shear gauge where we set $\chi \equiv 0$.
- φ in the zero-shear gauge ($\chi \equiv 0$) is *the same* as a gauge-invariant combination φ_χ.

Temporally gauge-invariant combinations:

$$\delta\mu_v \equiv \delta\mu - \dot{\mu}av, \quad \varphi_\chi \equiv \varphi - H\chi, \quad v_\chi \equiv v - \frac{1}{a}\chi,$$
$$\varphi_v \equiv \varphi - aHv, \quad \varphi_{\delta\phi} \equiv \varphi - \frac{H}{\dot{\phi}}\delta\phi \equiv -\frac{H}{\dot{\phi}}\delta\phi_\varphi, \quad \ldots \tag{146}$$

These are completely (i.e., both spatially and temporally) gauge-invariant.

> *"Many gauge-invariant combinations of these scalars can be constructed, but for the most part they have no physical meaning independent of a particular time gauge, or hypersurface condition."*
>
> J. M. Bardeen (1988)

Compared with notation in Bardeen (1980):

$$\epsilon_m \equiv \delta\mu_v/\mu, \quad \Phi_H \equiv \varphi_\chi, \quad v_s^{(0)} \equiv kv_\chi, \quad p\pi_L^{(0)} \equiv \delta p, \quad p\pi_T^{(0)} \equiv -\frac{\Delta}{a^2}\Pi. \tag{147}$$

Spatial gradient variable: (Olson 1976; Ellis and Bruni 1989; Woszczyna and Kułak 1989) [52, 35]

$$\tilde{h}^b_\alpha \tilde{\mu}_{,b} = \left(\tilde{\delta}^b_\alpha + \tilde{u}^b \tilde{u}_\alpha\right) \tilde{\mu}_{,b} = \delta\mu_{,\alpha} + \mu_{,0}\tilde{u}^0 \tilde{u}_\alpha = \left(\delta\mu - \dot{\mu} a v\right)_{,\alpha} + \dot{\mu} a v^{(v)}_\alpha$$
$$= \delta\mu_{v,\alpha} + \dot{\mu} a v^{(v)}_\alpha,$$

$$\tilde{h}^b_\alpha \tilde{\theta}_{,b} = \delta\theta_{v,\alpha} + \dot{\theta} a v^{(v)}_\alpha, \quad (148)$$

where, in the energy frame (91,93,90)

$$\tilde{u}^0 \equiv \frac{1}{a}(1-\alpha), \quad \tilde{u}_\alpha \equiv a(-v_{,\alpha} + v^{(v)}_\alpha). \quad (149)$$

The spatial gradient variable $\tilde{h}^b_a \tilde{\mu}_{,b}$ (or $\tilde{h}^b_a \tilde{\theta}_{,b}$) is a combination of $\delta\mu_v$ (or $\delta\theta_v = -\kappa_v$) and the rotational perturbation.

The gauge-invariant combinations $\delta\mu_v$ and $\delta\theta_v$ can be regarded as spatial gradients of density and expansion rate.

Gauge-ready method:

• There exist several (in fact, infinite number of) temporal gauge conditions available, and all of which have corresponding gauge-invariant counterpart. For example, for $\delta\mu$ we have:

$$\delta\mu_v, \quad \delta\mu_\varphi, \quad \delta\mu_\kappa, \quad \delta\mu_\chi, \quad \delta\mu_{\delta\mu} \equiv 0, \quad \dots . \tag{150}$$

> *"While a useful tool, gauge-invariance in itself does not remove all ambiguity in physical interpretation, ..."*
>
> J. M. Bardeen (1988)

• Often, mixed usage of different gauge invariant combinations is needed.
• Use the available temporal gauge conditions as the advantage.

> *"The moral is that one should work in the gauge that is mathematically most convenient for the problem at hand."*
>
> J. M. Bardeen (1988)

• Start without fixing the temporal gauge condition.
• Design equations for easy implentation of gauge conditions.
• Our notation of the gauge invariant combination shows systematically which gauge-invariant combination we are using, and allows algebra among different gauge-invariant combinations:

$$\delta\phi_\varphi \equiv \delta\phi - \frac{\dot{\phi}}{H}\varphi \equiv -\frac{\dot{\phi}}{H}\varphi_{\delta\phi}, \quad \varphi_v \equiv \varphi - aHv \equiv -aHv_\varphi, \quad \dots \tag{151}$$

Applications:

- Formulation [13] ← Lecture 2
- Ideal fluid [16], Zero-pressure fluid [18, 24] ← Lecture 2
- Scalar field [19, 17] ← Lecture 3
- Anisotropic universe [49]
- Coherent scalar field (axion) [21]
- CMB anisotropy (geodesic) [25, 34] ← Lecture 3
- CMB anisotropy (kinetic theory), collisionless particle [28]
- Multiple fluids and fields [26] ← Lecture 3
- Generalized gravity [12, 13, 20, 22, 31] ← Lecture 3
- Gravitational wave in generalized gravity [23]
- String theory correction terms [27, 31] ← Lecture 4
- Tachyonic correction term [30, 31] ← Lecture 4
- $\tilde{R}^{ab}\tilde{R}_{ab}$ gravity [50]
- Through non-singular bounce [29]
- Second-order perturbations [51, 32] ← Lecture 4
- Third-order perturbations [33] ← Lecture 4

4. Hydrodynamic Perturbations

From (98,99), (99,102,103), (101,103), (100,103), (99), (101) and (97,99,101) we can derive:

$$\frac{\Delta + 3K}{a^2}\varphi_\chi = -4\pi G\delta\mu_v, \tag{152}$$

$$\delta\dot{\mu}_v + 3H\delta\mu_v = \frac{\Delta + 3K}{a^2}\left[a(\mu + p)v_\chi + 2H\Pi\right], \tag{153}$$

$$\dot{v}_\chi + Hv_\chi = \frac{1}{a}\left(\alpha_\chi + \frac{\delta p_v}{\mu + p} + \frac{2}{3}\frac{\Delta + 3K}{a^2}\frac{\Pi}{\mu + p}\right), \tag{154}$$

$$\dot{\kappa}_v + 2H\kappa_v = 4\pi G\delta\mu_v + \frac{1}{\mu + p}\frac{\Delta + 3K}{a^2}\left[\delta p_v + \frac{2}{3}\left(3\dot{H} + \frac{\Delta}{a^2}\right)\Pi\right], \tag{155}$$

$$\kappa_v = \frac{\Delta + 3K}{a}v_\chi, \tag{156}$$

$$\varphi_\chi + \alpha_\chi = -8\pi G\Pi, \tag{157}$$

$$\dot{\varphi}_\chi + H\varphi_\chi = -4\pi G(\mu + p)av_\chi - 8\pi GH\Pi. \tag{158}$$

Notice the mixed usage of different gauge conditions. (Bardeen 1980)

(152) ~ Poisson's equation
(153) ~ Mass conservation (Continuity) equation
(154), (155) ~ Momentum conservation (Euler) equation

Correspondences:

$\delta\mu_v$, φ_χ, v_χ (κ_v) ~ Newtonian $\delta\varrho$, $\delta\Phi$, $\mathbf{u}$

Derivation of (153,156):

(102) in the comoving gauge gives:

$$\delta\dot{\mu}_v + 3H\left(\delta\mu_v + \delta p_v\right) = (\mu + p)\left(\kappa_v - 3H\alpha_v\right). \tag{159}$$

(99) in the comoving gauge gives

$$\kappa_v = -\frac{\Delta + 3K}{a^2}\chi_v = -\frac{\Delta + 3K}{a^2}\left(\chi - av\right) = \frac{\Delta + 3K}{a}v_\chi. \tag{160}$$

This gives (156).
(103) in the comoving gauge gives

$$\alpha_v = -\frac{1}{\mu + p}\left(\delta p_v + \frac{2}{3}\frac{\Delta + 3K}{a^2}\Pi\right). \tag{161}$$

Combing these equations give (153).

Density fluctuation:

From (152-154) we can derive ($c_s^2 \equiv \frac{\dot{p}}{\dot{\mu}}$ and $w \equiv \frac{p}{\mu}$) (Nariai 1969; Bardeen 1980):

$$\ddot{\delta}_v + (2 + 3c_s^2 - 6w)H\dot{\delta}_v + \left[- c_s^2\frac{\Delta}{a^2} - 4\pi G\mu(1 - 6c_s^2 + 8w - 3w^2) + 12(w - c_s^2)\frac{K}{a^2} + (3c_s^2 - 5w)\Lambda \right] \delta_v = \text{stresses}. \tag{162}$$

This can be written in a compact form for general K, Λ, and $p(\mu)$ [24]:

$$\frac{1+w}{a^2 H}\left[\frac{H^2}{a(\mu+p)}\left(\frac{a^3\mu}{H}\delta_v\right)^{\cdot}\right]^{\cdot} - c_s^2\frac{\Delta}{a^2}\delta_v \; = \; \text{stresses}. \tag{163}$$

In super-sound-horizon scale without stresses we have a general solution:

$$\delta_v(\mathbf{x}, t) \propto \frac{H}{a^3\mu}\left[C(\mathbf{x})\int_0^t \frac{a(\mu+p)}{H^2}dt + d(\mathbf{x})\right]. \tag{164}$$

- $C(\mathbf{x})$ and $d(\mathbf{x})$ are two integration constants.
- C and d are relatively growing and decaying solutions in expanding phase, respectively.
- $C(\mathbf{x})$ encodes the spatial structures of the relatively growing solution.

Newtonian vs. Relativistic:

(66,162) give:

$$\ddot{\delta} + 2H\dot{\delta} + \left[- v_s^2 \frac{\Delta}{a^2} - 4\pi G \varrho \right] \delta = 0, \tag{165}$$

$$\ddot{\delta}_v + (2 + 3c_s^2 - 6w)H\dot{\delta}_v + \left[- c_s^2 \frac{\Delta}{a^2} - 4\pi G\mu(1 - 6c_s^2 + 8w - 3w^2) + 12(w - c_s^2)\frac{K}{a^2} \right.$$
$$\left. + (3c_s^2 - 5w)\Lambda \right] \delta_v = 0. \tag{166}$$

Coincide in the zero-pressure case.

Compact form:

$$\frac{1+w}{a^2 H} \left[\frac{H^2}{a(\mu + p)} \left(\frac{a^3 \mu}{H} \delta_v \right)^{\cdot} \right]^{\cdot} - c_s^2 \frac{\Delta}{a^2} \delta_v = 0. \tag{167}$$

Valid for general K, Λ, and time varying $p = p(\mu)$.

Incorrect one in the synchronous gauge ($\alpha \equiv 0$) (for $w = \text{const.}$, $K = 0 = \Lambda$):

$$\ddot{\delta} + 2H\dot{\delta} + \left[- c_s^2 \frac{\Delta}{a^2} - 4\pi G\mu(1 + w)(1 + 3w) \right] \delta = 0. \tag{168}$$

Weinberg (72), Peebles (93), Coles-Lucchin (95,02), Moss (96), Padmanabhan (96), Longair (98), Peacock (99), …
Apparently, this is a popular error in textbooks. For corrections see [15, 24].

Curvature fluctuations:

For $K = 0$, we can show (next page) [31]:

$$\varphi_v = \frac{H^2}{4\pi G(\mu + p)a}\left(\frac{a}{H}\varphi_\chi\right)^{\cdot} + 2H^2\frac{\Pi}{\mu + p}, \tag{169}$$

$$\dot{\varphi}_v = \frac{H c_s^2 \Delta}{4\pi G(\mu + p)a^2}\varphi_\chi - \frac{H}{\mu + p}\left(e + \frac{2}{3}\frac{\Delta}{a^2}\Pi\right), \tag{170}$$

where $\delta p \equiv c_s^2 \delta\mu + e$.

Ideal fluid:

We have $e \equiv 0 \equiv \Pi$, thus

$$\varphi_v = \frac{H^2}{4\pi G(\mu + p)a}\left(\frac{a}{H}\varphi_\chi\right)^{\cdot}, \quad \dot{\varphi}_v = \frac{H c_s^2 \Delta}{4\pi G(\mu + p)a^2}\varphi_\chi. \tag{171}$$

Scalar field:

We have (next page) $e = -\frac{1-c_s^2}{4\pi G}\frac{\Delta}{a^2}\varphi_\chi$ and $\Pi = 0$, thus,

$$\varphi_v = \frac{H^2}{4\pi G(\mu + p)a}\left(\frac{a}{H}\varphi_\chi\right)^{\cdot}, \quad \dot{\varphi}_v = \frac{H \Delta}{4\pi G(\mu + p)a^2}\varphi_\chi. \tag{172}$$

Thus, in the case of a field, simply set $c_s^2 \to 1$.

Derivation of (169,170):

We have:

$$\varphi_v \equiv \varphi - aHv = \varphi_\chi - aHv_\chi = \varphi_\chi + \frac{H}{4\pi G(\mu+p)}\left(\dot{\varphi}_\chi + H\varphi_\chi + 8\pi GH\Pi\right)$$
$$= \frac{H^2}{4\pi G(\mu+p)a}\left(\frac{a}{H}\varphi_\chi\right)^{\cdot} + 2H^2\frac{\Pi}{\mu+p}, \qquad (173)$$

where we used (158) and background equation with $K = 0$.

We have:

$$\dot{\varphi}_v \equiv (\varphi - aHv)^{\cdot} = (\varphi_\chi - aHv_\chi)^{\cdot} = \dot{\varphi}_\chi - aH\left[\dot{v}_\chi + \left(H + \frac{\dot{H}}{H}\right)v_\chi\right]. \qquad (174)$$

Using (152,157,154,158) we can show (170).

Derivation of (172):

A minimally coupled scalar field can be regarded as a fluid with the fluid quantities in (116):

$$\mu = \frac{1}{2}\dot{\phi}^2 + V, \quad p = \frac{1}{2}\dot{\phi}^2 - V,$$

$$\delta\mu = \dot{\phi}\delta\dot{\phi} - \dot{\phi}^2\alpha + V_{,\phi}\delta\phi, \quad \delta p = \dot{\phi}\delta\dot{\phi} - \dot{\phi}^2\alpha - V_{,\phi}\delta\phi, \quad (\mu + p)v = \frac{1}{a}\dot{\phi}\delta\phi, \quad \Pi = 0.$$

We have

$$\mu + p = \dot{\phi}^2, \quad w \equiv \frac{p}{\mu} = \frac{\frac{1}{2}\dot{\phi}^2 - V}{\frac{1}{2}\dot{\phi}^2 + V}, \quad c_s^2 \equiv \frac{\dot{p}}{\dot{\mu}} = \frac{\ddot{\phi} - V_{,\phi}}{\ddot{\phi} + V_{,\phi}}. \tag{175}$$

Using the gauge-invariance of e we have:

$$e \equiv \delta p - c_s^2\delta\mu = \left(1 - c_s^2\right)\left(-\dot{\phi}^2\alpha_{\delta\phi}\right) = \left(1 - c_s^2\right)\delta\mu_{\delta\phi} = -\frac{1 - c_s^2}{4\pi G}\frac{\Delta}{a^2}\varphi_\chi. \tag{176}$$

In the last step we used $\delta\mu_{\delta\phi} = \delta\mu_v$ and (152).
Thus, eqs. (169,170) give :

$$\varphi_v = \frac{H^2}{4\pi G(\mu + p)a}\left(\frac{a}{H}\varphi_\chi\right)^{\cdot}, \quad \dot{\varphi}_v = \frac{H\Delta}{4\pi G(\mu + p)a^2}\varphi_\chi.$$

which is (172).

Equations in two gauges:

In an ideal fluid (169,170) give:

$$\varphi_v = \frac{H^2}{4\pi G(\mu+p)a}\left(\frac{a}{H}\varphi_\chi\right)^{\cdot}, \quad \dot{\varphi}_v = \frac{Hc_s^2\Delta}{4\pi G(\mu+p)a^2}\varphi_\chi. \tag{177}$$

Combining these (Field-Shepley 1968; Lukash 1980; Mukhanov 1985, 1988) [9, 42, 44, 45, 31]:

$$\frac{H^2c_s^2}{(\mu+p)a^3}\left[\frac{(\mu+p)a^3}{H^2c_s^2}\dot{\varphi}_v\right]^{\cdot} - c_s^2\frac{\Delta}{a^2}\varphi_v = \frac{Hc_s}{a^3\sqrt{\mu+p}}\left[v'' - \left(\frac{z''}{z} + c_s^2\Delta\right)v\right] = 0, \tag{178}$$

$$\frac{\mu+p}{H}\left[\frac{H^2}{(\mu+p)a}\left(\frac{a}{H}\varphi_\chi\right)^{\cdot}\right]^{\cdot} - c_s^2\frac{\Delta}{a^2}\varphi_\chi = \frac{\sqrt{\mu+p}}{a^2}\left[u'' - \left(\frac{(1/\bar{z})''}{1/\bar{z}} + c_s^2\Delta\right)u\right] = 0, \tag{179}$$

where

$$v \equiv z\varphi_v, \quad u \equiv \frac{1}{\bar{z}}\frac{a}{H}\varphi_\chi, \quad c_s z \equiv \frac{a\sqrt{\mu+p}}{H} \equiv \bar{z}. \tag{180}$$

Large-scale solutions:

$$\varphi_v = C - d\frac{k^2}{4\pi G}\int^\eta \frac{d\eta}{z^2}, \quad \varphi_\chi = 4\pi GC\frac{H}{a}\int^\eta \bar{z}^2 d\eta + d\frac{H}{a}. \tag{181}$$

Small-scale solutions:

$$v = z\Phi \propto e^{\pm ic_s k\eta}, \quad u = \frac{1}{\sqrt{\mu+p}}\varphi_\chi \propto e^{\pm ic_s k\eta}. \tag{182}$$

In the case of a field, simply set $c_s^2 \to 1$.

<u>**Exact solutions**</u> ($K = 0 = \Lambda$ w =constant) [51]:

$$a \propto t^{\frac{2}{3(1+w)}} \propto \eta^{\frac{2}{1+3w}}, \quad aH\eta = \frac{2}{1+3w}, \tag{183}$$

thus $z \propto \bar{z} \propto a$, and

$$\frac{z''}{z} = \frac{2(1-3w)}{(1+3w)^2}\frac{1}{\eta^2}, \quad \frac{(1/\bar{z})''}{(1/\bar{z})} = \frac{6(1+w)}{(1+3w)^2}\frac{1}{\eta^2}. \tag{184}$$

Thus

$$\varphi_v = \frac{v}{z} \equiv c_1(k)\frac{J_\nu(x)}{x^\nu} + c_2(k)\frac{Y_\nu(x)}{x^\nu}, \tag{185}$$

$$\varphi_\chi = \sqrt{\mu + p}u = \frac{3(1+w)}{1+3w}\left(c_1(k)\frac{J_{\bar{\nu}}(x)}{x^{\bar{\nu}}} + c_2(k)\frac{Y_{\bar{\nu}}(x)}{x^{\bar{\nu}}}\right), \tag{186}$$

where

$$x \equiv c_s k|\eta|, \quad \nu \equiv \frac{3(1-w)}{2(1+3w)}, \quad \bar{\nu} \equiv \nu + 1 = \frac{5+3w}{2(1+3w)}. \tag{187}$$

(152) gives

$$\delta_v = \frac{(1+3w)^2}{6w}x^2\varphi_\chi. \tag{188}$$

<u>**In the large-scale limit**</u> ($x \ll 1$) we have

$$\varphi_v \propto C, \; da^{-\frac{3}{2}(1-w)}, \tag{189}$$

$$\varphi_\chi \propto C, \; da^{-\frac{5+3w}{2}}, \tag{190}$$

$$\delta_v \propto Ca^{1+3w}, \; da^{-\frac{3}{2}(1-w)} \propto Ct^{\frac{2(1+3w)}{3(1+w)}}, \; dt^{-\frac{1-w}{1+w}} \propto C\eta^2, \; d\eta^{-\frac{3(1-w)}{1+3w}}. \tag{191}$$

The well known solutions in the matter ($w = 0$) and radiation ($w = \frac{1}{3}$) eras:

$$\begin{aligned} \text{mde}: \quad & \delta_v \propto Ca, \; da^{-\frac{3}{2}} \propto Ct^{\frac{2}{3}}, \; dt^{-1} \propto C\eta^2, \; d\eta^{-3}, \\ \text{rde}: \quad & \delta_v \propto Ca^2, \; da^{-1} \propto Ct, \; dt^{-\frac{1}{2}} \propto C\eta^2, \; d\eta^{-1}. \end{aligned} \tag{192}$$

If we consider only the C-mode which is the relatively growing-mode in an expanding phase:

$$\varphi_v(\mathbf{x}, t) = C(\mathbf{x}), \tag{193}$$

$$\varphi_\chi(\mathbf{x}, t) = \frac{3+3w}{5+3w} C(\mathbf{x}). \tag{194}$$

<u>$C(\mathbf{x})$:</u>

- Integration constant of the growing mode.
- Every variable contains it.
- Characterizes the large scale evolution.
- Encodes the spatial structure which is preserved.

Comparison with other notations:

$\varphi_\chi =$	Φ_H	Bardeen (1980)
$\varphi_\delta =$	ζ	Bardeen (1988)
$\alpha_\chi =$	Φ_A	Bardeen (1980)
	Φ	Mukhanov *et al* (1992)
$\varphi_\chi =$	Φ_H	Bardeen (1980)
	$-\Psi$	Mukhanov *et al* (1992)
$\varphi_v =$	ϕ_m	Bardeen (1980)
	$\mathcal{R}$	Liddle and Lyth (2000)

"*The advantages of Φ_H and Φ_A as variables are the advantages of working in the zero-shear gauge, no more and no less, which . . . are not overwhelming.*"

J. M. Bardeen (1988)

5. Covariant approach

In the energy frame, $\tilde{q}_a \equiv 0$, (20,18,12,19) give [14]:

$$\tilde{H}^2 = \frac{8\pi G}{3}\tilde{\mu} - \frac{\tilde{K}}{a^2} + \frac{\Lambda}{3} + \frac{1}{3}\tilde{\sigma}^2, \tag{195}$$

$$\dot{\tilde{H}} + \tilde{H}^2 = -\frac{4\pi G}{3}\left(\tilde{\mu} + 3\tilde{p}\right) + \frac{\Lambda}{3} + \frac{1}{3}\tilde{\nabla}_a \tilde{a}^a - \frac{2}{3}\left(\tilde{\sigma}^2 - \tilde{\omega}^2\right), \tag{196}$$

$$\dot{\tilde{\mu}} = -3\tilde{H}\left(\tilde{\mu} + \tilde{p}\right) - \tilde{\pi}^{ab}\tilde{\sigma}_{ab}, \tag{197}$$

$$\tilde{a}_a = -\frac{1}{\tilde{\mu} + \tilde{p}}\tilde{h}^b_a\left(\tilde{\nabla}_b\tilde{p} + \tilde{\nabla}_c\tilde{\pi}^c_b\right), \tag{198}$$

where we introduced

$$3\tilde{H} \equiv \tilde{\theta} \equiv \tilde{\nabla}_a \tilde{u}^a, \quad \tilde{R}^{(3)} \equiv \frac{6\tilde{K}}{a^2}. \tag{199}$$

In Friedmann background:

By simply imposing the symmerty:

$$\tilde{H}^2 = \frac{8\pi G}{3}\tilde{\mu} - \frac{\tilde{K}}{a^2} + \frac{\Lambda}{3}, \tag{200}$$

$$\dot{\tilde{H}} + \tilde{H}^2 = -\frac{4\pi G}{3}\left(\tilde{\mu} + 3\tilde{p}\right) + \frac{\Lambda}{3}, \tag{201}$$

$$\dot{\tilde{\mu}} = -3\tilde{H}\left(\tilde{\mu} + \tilde{p}\right). \tag{202}$$

Perhaps, this is the simplest way to derive these cosmological equations.

To linear order in Friedmann world model:

(195-197) give:

$$\tilde{H}^2 = \frac{8\pi G}{3}\tilde{\mu} - \frac{\tilde{K}}{a^2} + \frac{\Lambda}{3}, \tag{203}$$

$$\dot{\tilde{\tilde{H}}} + \tilde{H}^2 = -\frac{4\pi G}{3}\left(\tilde{\mu} + 3\tilde{p}\right) + \frac{\Lambda}{3} + \frac{1}{3}\tilde{\nabla}_a \tilde{a}^a, \tag{204}$$

$$\dot{\tilde{\tilde{\mu}}} = -3\tilde{H}\left(\tilde{\mu} + \tilde{p}\right). \tag{205}$$

Combing these:

$$\frac{\dot{\tilde{\tilde{K}}}}{a^2} = -\frac{2}{3}\tilde{H}\tilde{\nabla}_a \tilde{a}^a - 2\left(\tilde{H} - \frac{\dot{a}}{a}\right)\frac{\tilde{K}}{a^2}. \tag{206}$$

Since (195) is valid in the normal-frame $\tilde{n}_\alpha = 0$, together with $\tilde{q}_a = 0$, it implies that we are already in the comoving gauge.

In the large scale and near flat case

$$\dot{\tilde{\tilde{K}}} = 0 \quad \rightarrow \quad \tilde{K} = \tilde{K}(\mathbf{x}). \tag{207}$$

Thus

$$\tilde{H}^2 = \frac{8\pi G}{3}\tilde{\mu} - \frac{\tilde{K}(\mathbf{x})}{a^2} + \frac{\Lambda}{3}. \tag{208}$$

In the large-scale, perturbed Friedmann universe evolves like Friedmann background with perturbed $\tilde{K}(\mathbf{x})$ [43, 14].

Fluid flow approach:

Now we perturb (195-198). We take the comoving gauge.
Using (149)

$$\dot{\tilde{H}} \equiv \tilde{H}_{,a}\tilde{u}^a = \tilde{H}_{,0}\tilde{u}^0 + \tilde{H}_{,\alpha}\tilde{u}^\alpha = (H+\delta H)_{,0}\frac{1}{a}(1-\alpha) = \dot{H} + \delta\dot{H} - \dot{H}\alpha. \tag{209}$$

Proper-time derivative along $\tilde{u}^a$ is related to coordinate-time derivative corrected by lapse-function α which is related to pressures: (103) in the comoving gauge condition gives:

$$\alpha = -\frac{1}{\mu+p}\left(\delta p + \frac{2}{3}\frac{\Delta+3K}{a^2}\Pi\right). \tag{210}$$

Ignoring the anisotropic stress (195-197) give:

$$2H\delta H = \frac{8\pi G}{3}\delta\mu - \frac{\delta K}{a^2}, \tag{211}$$

$$\delta\dot{H} + 2H\delta H = -\frac{4\pi G}{3}(\delta\mu + 3\delta p) - \frac{1}{\mu+p}\left(\frac{\Delta}{3a^2} + \dot{H}\right)\delta p, \tag{212}$$

$$\delta\dot{\mu} + 3H\delta\mu = -3(\mu+p)\delta H. \tag{213}$$

Combining (212,213) we can derive equation for $\ddot{\delta}_v$ in (162).
Combining (211-213):

$$\delta\dot{K} = \frac{2}{3}(\Delta + 3K)\frac{H\delta p}{\mu+p} - 2K\delta H. \tag{214}$$

In large scales where the pressure gradient term can be neglected, and in a near flat model:

$$\delta K(\mathbf{x}, t) = \text{constant}. \tag{215}$$

Lecture 3

1. **Scalar field perturbation**
2. **Quantum generation**
3. **Inflationary spectra**
4. **CMB anisotropies**
5. **Multi-component case**

"The universe was brought into being in a less than fully formed state, but was gifted with the capacity to transform itself from unformed matter into a truly marvellous array of structure and life forms."

Saint Augustine (354-430)

1. Minimally coupled scalar field

Derivation: [17]

In the uniform-curvature gauge $\varphi \equiv 0$ (thus $\delta\phi = \delta\phi_\varphi$, etc), assuming $K = 0$, (115) give:

$$\delta\ddot{\phi}_\varphi + 3H\delta\dot{\phi}_\varphi + \left[-\frac{\Delta}{a^2} + V_{,\phi\phi}\right]\delta\phi_\varphi = \underbrace{\dot{\phi}\left(\kappa_\varphi + \dot{\alpha}_\varphi\right) + \left(2\ddot{\phi} + 3H\dot{\phi}\right)\alpha_\varphi}_{\text{from metric fluctuation}}. \tag{216}$$

(97,99,116), (98,116) give the metric perturbation in terms of the field fluctuations:

$$\alpha_\varphi = \frac{4\pi G}{H}\dot{\phi}\delta\phi_\varphi, \tag{217}$$

$$\kappa_\varphi = -\frac{4\pi G}{H}\left(\dot{\phi}\delta\dot{\phi}_\varphi + \frac{\dot{H}}{H}\dot{\phi}\delta\phi_\varphi + V_{,\phi}\delta\phi_\varphi\right). \tag{218}$$

Combining these:

$$\delta\ddot{\phi}_\varphi + 3H\delta\dot{\phi}_\varphi + \left[-\frac{\Delta}{a^2} + V_{,\phi\phi} \underbrace{+ 2\frac{\dot{H}}{H}\left(3H - \frac{\dot{H}}{H} + 2\frac{\ddot{\phi}}{\dot{\phi}}\right)}_{\text{from metric fluctuation}}\right]\delta\phi_\varphi = 0. \tag{219}$$

Simplest compared with equations in other gauge conditions [19].

Compared with quantum field in curved space:

Equation: [17]

$$\underbrace{\delta\ddot{\phi}_\varphi + 3H\delta\dot{\phi}_\varphi + \left[-\frac{\Delta}{a^2} + V_{,\phi\phi}\right.}_{\text{without metric pert.}} \underbrace{\left. +2\frac{\dot{H}}{H}\left(3H - \frac{\dot{H}}{H} + 2\frac{\ddot{\phi}}{\dot{\phi}}\right)\right]}_{\text{from metric fluctuation}} \delta\phi_\varphi = 0, \tag{220}$$

$$\underbrace{\ddot{\phi} + 3H\dot{\phi} - \frac{\Delta}{a^2}\phi + V_{,\phi}}_{\text{quantum field in curved space}} = 0. \tag{221}$$

Exponential $a \propto e^{Ht}$, or Power-law $a \propto t^p$ expansions:

$$\delta\ddot{\phi}_\varphi + 3H\delta\dot{\phi}_\varphi - \frac{\Delta}{a^2}\delta\phi_\varphi = 0 \quad \Leftrightarrow \quad \text{QFCS}. \tag{222}$$

Compact form:

$$\frac{H}{a^3\dot{\phi}}\left[\frac{a^3\dot{\phi}^2}{H^2}\left(\frac{H}{\dot{\phi}}\delta\phi_\varphi\right)^{\cdot}\right]^{\cdot} - \frac{\Delta}{a^2}\delta\phi_\varphi = 0. \tag{223}$$

Large-scale general solution:

$$\varphi_{\delta\phi} = -\frac{H}{\dot{\phi}}\delta\phi_\varphi = C(\mathbf{x}) - \underbrace{D(\mathbf{x})\int_0^t \frac{H^2}{a^3\dot{\phi}^2}dt}_{\text{transient}}. \tag{224}$$

Proper choice of the gauge is important!

Correspondences: [24, 17]

$$\delta\mu_v,\ \varphi_\chi,\ v_\chi\ (\kappa_v) \sim \text{Newtonian } \delta\varrho,\ \delta\Phi,\ \mathbf{u}$$

$$\delta\phi_\varphi \sim \phi \text{ in QFCS}$$

$$\varphi_v = \varphi_{\delta\phi} \sim \text{super-sound-horizon conservation}$$

$\delta\mu_v \equiv \delta_v - \dot{\mu} a v =$ perturbed density $\delta\mu$ in the comoving gauge ($v \equiv 0$).
$\varphi_\chi \equiv \varphi - H\chi =$ perturbed three-space curvature φ in the zero-shear gauge ($\chi \equiv 0$).
$v_\chi \equiv v - \frac{1}{a}\chi =$ perturbed velocity v in the zero-shear gauge ($\chi \equiv 0$).
$\kappa_v \equiv \kappa + (3\dot{H} + \frac{\Delta}{a^2}) a v =$ perturbed expansion scalar κ in the comoving gauge ($v \equiv 0$).
$\varphi_v \equiv \varphi - aHv =$ perturbed three-space curvature φ in the comoving gauge ($v \equiv 0$).
$\varphi_{\delta\phi} \equiv \varphi - \frac{H}{\dot{\phi}}\delta\phi =$ perturbed three-space curvature φ in the uniform-field gauge ($\delta\phi \equiv 0$).
$\delta\phi_\varphi \equiv \delta\phi - \frac{\dot{\phi}}{H}\varphi =$ perturbed scalar field $\delta\phi$ in the uniform-curvature gauge ($\varphi \equiv 0$).

These are all gauge invariant.

2. Quantum Generation: [17]

Action:

$$\tilde{S} = \int \left[\frac{1}{16\pi G}\tilde{R} - \frac{1}{2}\tilde{\phi}^{,a}\tilde{\phi}_{,a} - V(\tilde{\phi})\right]\sqrt{-\tilde{g}}d^4x. \tag{225}$$

Perturbed action: (Mukhanov 1988)

$$\delta^2 S = \frac{1}{2}\int a^3\left\{\delta\dot{\phi}_\varphi^2 - \frac{1}{a^2}\delta\phi_\varphi{}^{,\alpha}\delta\phi_{\varphi,\alpha} + \frac{H}{a^3\dot{\phi}}\left[a^3\left(\frac{\dot{\phi}}{H}\right)^{\cdot}\right]^{\cdot}\delta\phi_\varphi^2\right\}dtd^3x. \tag{226}$$

Semiclassical decomposition:

$$\tilde{\phi}(\mathbf{x},t) \equiv \phi(t) + \delta\hat{\phi}(\mathbf{x},t), \quad \delta\hat{\phi}_\varphi \equiv \delta\hat{\phi} - \frac{\dot{\phi}}{H}\hat{\varphi}. \tag{227}$$

Mode expansion:

$$\delta\hat{\phi}_\varphi(\mathbf{x},t) \equiv \int \frac{d^3k}{(2\pi)^{3/2}}\left[\hat{a}_\mathbf{k}\delta\phi_\mathbf{k}(t)e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{a}_\mathbf{k}^\dagger\delta\phi_\mathbf{k}^*(t)e^{-i\mathbf{k}\cdot\mathbf{x}}\right], \tag{228}$$

$$[\hat{a}_\mathbf{k}, \hat{a}_{\mathbf{k}'}] = 0, \quad [\hat{a}_\mathbf{k}^\dagger, \hat{a}_{\mathbf{k}'}^\dagger] = 0, \quad [\hat{a}_\mathbf{k}, \hat{a}_{\mathbf{k}'}^\dagger] = \delta^3(\mathbf{k} - \mathbf{k}'). \tag{229}$$

Mode evolution equation:

$$\delta\ddot{\phi}_\mathbf{k} + 3H\delta\dot{\phi}_\mathbf{k} + \left[\frac{k^2}{a^2} + V_{,\phi\phi} + 2\frac{\dot{H}}{H}\left(3H - \frac{\dot{H}}{H} + 2\frac{\ddot{\phi}}{\dot{\phi}}\right)\right]\delta\phi_\mathbf{k} = 0. \tag{230}$$

Equal-time commutation relation:

$$[\delta\hat{\phi}(\mathbf{x},t),\delta\hat{\pi}(\mathbf{x}',t)] \equiv i\delta^3(\mathbf{x}-\mathbf{x}'), \quad \delta\pi \equiv \partial\mathcal{L}/(\partial\delta\dot{\phi}) = a^3\delta\dot{\phi}, \tag{231}$$

$$\delta\phi_{\mathbf{k}}\delta\dot{\phi}^*_{\mathbf{k}} - \delta\phi^*_{\mathbf{k}}\delta\dot{\phi}_{\mathbf{k}} = ia^{-3}. \tag{232}$$

Power spectrum:

Vacuum expectation:

$$\mathcal{P}_{\delta\hat{\phi}}(k,t) \equiv \frac{k^3}{2\pi^2}\int \langle\delta\hat{\phi}(\mathbf{x}+\mathbf{r},t)\delta\hat{\phi}(\mathbf{x},t)\rangle_{\rm vac} e^{-i\mathbf{k}\cdot\mathbf{r}}d^3r = \frac{k^3}{2\pi^2}\left|\delta\phi_{\mathbf{k}}(t)\right|^2. \tag{233}$$

Spatial average:

$$\mathcal{P}_{\delta\phi}(k,t) \equiv \frac{k^3}{2\pi^2}\int \langle\delta\phi(\mathbf{x}+\mathbf{r},t)\delta\phi(\mathbf{x},t)\rangle_{\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{r}}d^3r = \frac{k^3}{2\pi^2}\left|\delta\phi(\mathbf{k},t)\right|^2, \tag{234}$$

where $\langle\rangle_{\rm vac} \equiv \langle{\rm vac}||{\rm vac}\rangle$ with $a_{\mathbf{k}}|{\rm vac}\rangle \equiv 0$ for every $\mathbf{k}$, and, $\langle f\rangle_{\mathbf{x}} \equiv \int f d^3x/\int d^3x$.

Ansatz:

$$\mathcal{P}_{\delta\hat{\phi}}(k,t) \Leftrightarrow \mathcal{P}_{\delta\phi}(k,t). \tag{235}$$

Spectral index:

$$\mathcal{P}_{\varphi_v} \propto k^{ns-1}, \tag{236}$$

where $\varphi_v = \varphi_{\delta\phi} = -\frac{H}{\dot{\phi}}\delta\phi_{\varphi}$, thus $\mathcal{P}_{\varphi_v} = \mathcal{P}_{\varphi_{\delta\phi}} = \left|\frac{H}{\dot{\phi}}\right|^2\mathcal{P}_{\delta\phi_{\varphi}}$.

Large-scale Structures from Quantum Fluctuations

Quantum Fluctuations

$\Downarrow$ $\leftarrow$ accelerated expansion

$\mathcal{P}_{\delta\hat{\phi}_\varphi}$

$\Downarrow$ $\leftarrow$ become macroscopic during acceleration phase, classicalize

$\mathcal{P}_{\delta\phi_\varphi}$

$\Downarrow$ $\leftarrow$ acceleration era ends $\quad \varphi_v = \varphi_{\delta\phi} = -\frac{H}{\dot{\phi}}\delta\phi_\varphi$

$\mathcal{P}_{\varphi_v}$

$\Downarrow$ $\leftarrow$ fluid eras begin

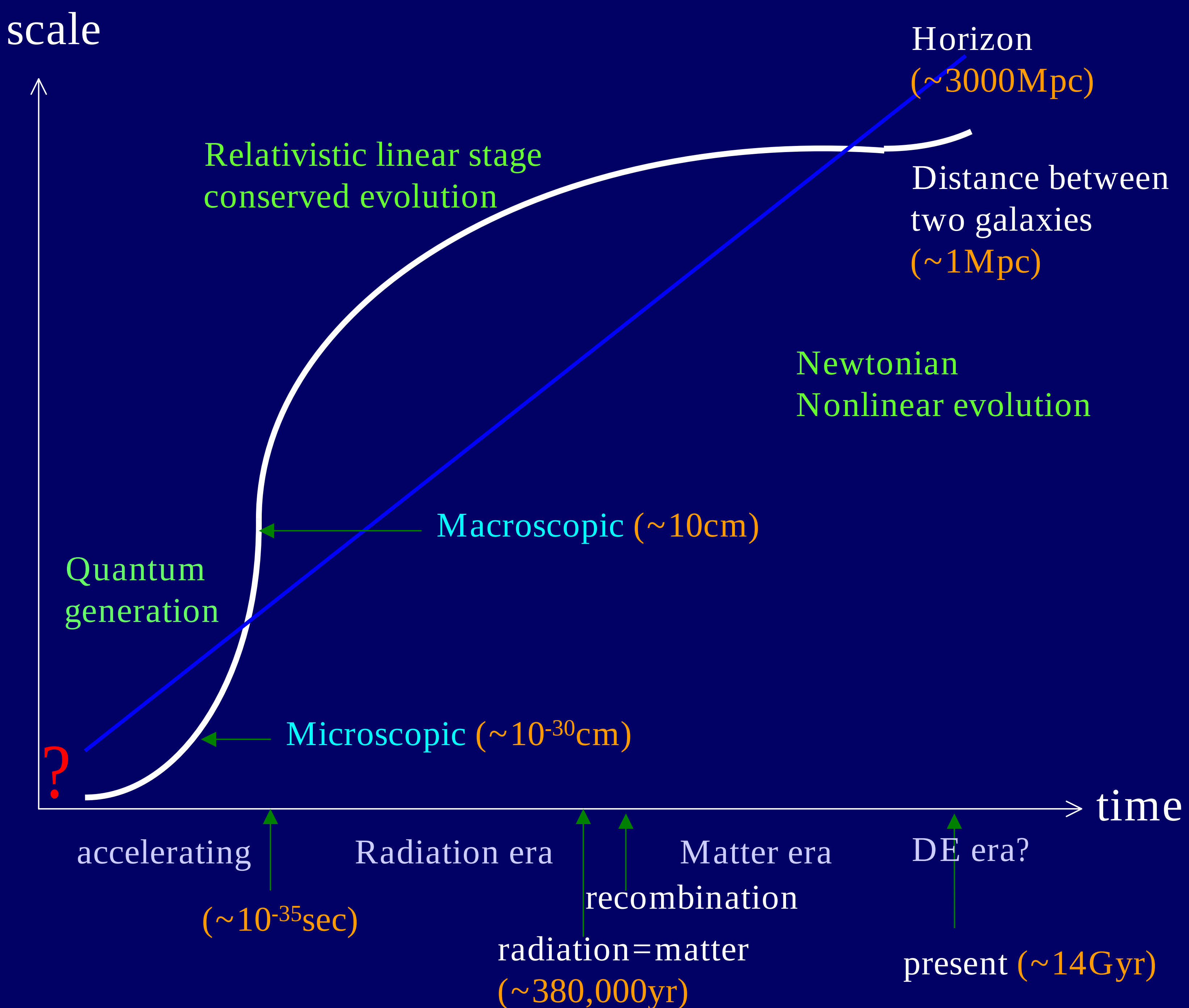
$\mathcal{P}_{\delta\Phi}, \mathcal{P}_{\delta\mu}, \mathcal{P}_{\delta v}, \ldots$ $\leftarrow$ follows from conserved $\mathcal{P}_{\varphi_v}$

$\Downarrow$

Fluctuations in CMB, Large-scale Structures

In MDE, $K = 0 = \Lambda$:

$$\varphi_v = C, \quad \delta\Phi = -\frac{3}{5}C, \quad \frac{\delta\varrho}{\varrho} = \frac{2}{5}\left(\frac{k}{aH}\right)^2 C, \quad \frac{\delta T}{T} = -\frac{1}{5}C.$$


scale
Horizon
(~3000Mpc)
Relativistic linear stage
conserved evolution
Distance between
two galaxies
(~1Mpc)
Newtonian
Nonlinear evolution
Macroscopic (~10cm)
Quantum
generation
Microscopic (~10^-30 cm)
?
time
accelerating
Radiation era
Matter era
DE era?
recombination
(~10^-35 sec)
radiation=matter
(~380,000yr)
present (~14Gyr)

3. Inflationary Spectra

Exponential expansion: [17]

Background:

$$a = a_0 e^{H(t-t_0)}, \quad H = \text{constant}, \quad \dot{\phi} = 0, \quad V = \text{constant}. \tag{237}$$

Equation:

$$\delta\ddot{\phi}_{\mathbf{k}} + 3H\delta\dot{\phi}_{\mathbf{k}} + \frac{k^2}{a^2}\delta\phi_{\mathbf{k}} = 0. \tag{238}$$

Solution:

$$\delta\phi_{\mathbf{k}}(t) = \frac{\sqrt{\pi}}{2} H\eta^{3/2}\left[c_1(k)H_\nu^{(1)}(k\eta) + c_2(k)H_\nu^{(2)}(k\eta)\right], \quad \nu \equiv \sqrt{\frac{9}{4} - \frac{m^2}{H^2}},$$
$$|c_2(k)|^2 - |c_1(k)|^2 = 1. \tag{239}$$

Large-scale power spectra:

$$\mathcal{P}^{1/2}_{\hat{\varphi}_{\delta\phi}}(k,t) = \frac{H^2}{2\pi|\dot{\phi}|}\,|c_2(k) - c_1(k)| \quad \propto \quad k^{n_S - 1}, \tag{240}$$

$$\mathcal{P}^{1/2}_{\hat{C}^{(t)}_{\alpha\beta}}(\mathbf{k},\eta) = \sqrt{16\pi G}\frac{H}{2\pi}\sqrt{\frac{1}{2}\sum_{\ell}\left|c_{\ell 2}(\mathbf{k}) - c_{\ell 1}(\mathbf{k})\right|^2} \quad \propto \quad k^{n_T}. \tag{241}$$

Bunch-Davies (adiabatic) vacuum:

$$c_2(k) \equiv 1, \quad c_1(k) \equiv 0. \tag{242}$$

Harrison-Zel'dovich spectrum

Simple vacuum choice $\Rightarrow$ $n_S \sim 1$, $n_T \sim 0$ $\Leftrightarrow$ CMB, LSS Observations.

Power-law expansion: [17]

Background with $a \propto t^{2/(3+3w)} \propto t^p$ with $w = \text{constant}$ [41]

$$V(\phi) = \frac{(1-w)}{12\pi G(1+w)^2} e^{-\sqrt{24\pi G(1+w)}\phi}, \quad \phi = \sqrt{\frac{1}{6\pi G(1+w)}} \ln t. \tag{243}$$

We have

$$V_{,\phi\phi} + 2\frac{\dot{H}}{H}\left(3H - \frac{\dot{H}}{H} + 2\frac{\ddot{\phi}}{\dot{\phi}}\right) = -\frac{H}{a^3\dot{\phi}}\left[a^3\left(\frac{\dot{\phi}}{H}\right)^{\cdot}\right]^{\cdot} = 0. \tag{244}$$

Equation:

$$\delta\ddot{\phi}_{\mathbf{k}} + 3H\delta\dot{\phi}_{\mathbf{k}} + \frac{k^2}{a^2}\delta\phi_{\mathbf{k}} = 0, \tag{245}$$

Solution:

$$\delta\phi_{\mathbf{k}}(t) = -\frac{\sqrt{\pi\eta}}{2a}\left[c_1(k)H_\nu^{(1)}(k\eta) + c_2(k)H_\nu^{(2)}(k\eta)\right], \quad \nu \equiv \frac{3(w-1)}{2(3w+1)} = \frac{3p-1}{2(p-1)}.$$

$$|c_2(k)|^2 - |c_1(k)|^2 = 1. \tag{246}$$

Large scale limit:

$$\mathcal{P}^{1/2}_{\delta\hat{\phi}_\varphi}(k,t) = \frac{\Gamma(\nu)}{\pi^{3/2}a|\eta|}\left(\frac{k|\eta|}{2}\right)^{3/2-\nu} |c_2(k) - c_1(k)| \quad \propto \quad k^{n_S-1}, \tag{247}$$

$$\mathcal{P}^{1/2}_{\hat{C}^{(t)}_{\alpha\beta}}(\mathbf{k},\eta) = \sqrt{16\pi G}\frac{H}{2\pi}\frac{\Gamma(\nu)}{\Gamma(3/2)}\frac{p-1}{p}\left(\frac{2}{k|\eta|}\right)^{\nu-3/2}\sqrt{\frac{1}{2}\sum_\ell \left|c_{\ell 2}(\mathbf{k}) - c_{\ell 1}(\mathbf{k})\right|^2} \propto k^{n_T}. \tag{248}$$

For large p and simple vacuum choice $\quad \Rightarrow \quad n_S \sim 1, \quad n_T \sim 0.$

Slow-roll inflation: [31]

Slow-roll parameters:

$$\epsilon_1 \equiv \frac{\dot{H}}{H^2}, \quad \epsilon_2 \equiv \frac{\ddot{\phi}}{H\dot{\phi}}. \tag{249}$$

For $\dot{\epsilon}_i = 0$ and $|\epsilon_i| \ll 1$: [57]

Power-spectra: ($\gamma_1 \equiv \gamma_E + \ln 2 - 2 = -0.7296\ldots$)

$$\mathcal{P}^{1/2}_{\hat{\varphi}_{\delta\phi}}\Big|_{LS} = \frac{H^2}{2\pi|\dot{\phi}|}\Big\{1 + \epsilon_1 + \big[\gamma_1 + \ln(k|\eta|)\big](2\epsilon_1 - \epsilon_2)\Big\} \quad \propto \quad k^{n_S-1}, \tag{250}$$

$$\mathcal{P}^{1/2}_{\hat{C}^{(t)}_{\alpha\beta}}\Big|_{LS} = \sqrt{16\pi G}\frac{H}{2\pi}\Big\{1 + \epsilon_1 + \big[\gamma_1 + \ln(k|\eta|)\big]\epsilon_1\Big\} \quad \propto \quad k^{n_T}. \tag{251}$$

Spectral indices:

$$n_S - 1 \equiv \frac{\partial \ln \mathcal{P}_{\varphi_v}}{\partial \ln k} = 2(2\epsilon_1 - \epsilon_2), \quad n_T \equiv \frac{\partial \ln \mathcal{P}_{C^{(t)}_{ab}}}{\partial \ln k} = 2\epsilon_1. \tag{252}$$

Classical spectra:

For Harrison-Zel'dovich ($n_S - 1 = 0 = n_T$) spectra with $K = 0 = \Lambda$:

$$\langle a_2^2 \rangle = \langle a_2^2 \rangle_S + \langle a_2^2 \rangle_T = \frac{\pi}{75}\mathcal{P}_{\varphi_{\delta\phi}} + 7.74\frac{1}{5}\frac{3}{32}\mathcal{P}_{C^{(t)}_{\alpha\beta}}. \tag{253}$$

Thus

$$r_2 \equiv \langle a_2^2 \rangle_T / \langle a_2^2 \rangle_S = 13.8|\epsilon_1| = 6.9 n_T. \tag{254}$$

Gravitational wave: [23]

For $K = 0$ we have:

$$\delta^2 S_g = \int \frac{1}{16\pi G} a^3 \left(\dot{C}^{(t)\alpha}_{\ \ \beta} \dot{C}^{(t)\beta}_{\ \ \alpha} - \frac{1}{a^2} C^{(t)\alpha}_{\ \ \beta,\gamma} C^{(t)\beta|\gamma}_{\ \ \alpha} \right) dt d^3x. \tag{255}$$

We consider Hilbert space operator $\hat{C}^{(t)}_{\alpha\beta}$ and expand [2]:

$$\hat{C}^{(t)}_{\alpha\beta}(\mathbf{x}, t) \equiv \int \frac{d^3k}{(2\pi)^{3/2}} \hat{C}^{(t)}_{\alpha\beta}(\mathbf{x}, t; \mathbf{k}) \equiv \int \frac{d^3k}{(2\pi)^{3/2}} \left[\sum_\ell e^{i\mathbf{k}\cdot\mathbf{x}} \tilde{h}_{\ell\mathbf{k}}(t) \hat{a}_{\ell\mathbf{k}} e^{(\ell)}_{\alpha\beta}(\mathbf{k}) + \text{h.c.} \right],$$

$$[\hat{a}_{\ell\mathbf{k}}, \hat{a}^\dagger_{\ell'\mathbf{k}'}] = \delta_{\ell\ell'} \delta^3(\mathbf{k} - \mathbf{k}'), \quad \text{zero otherwise}, \tag{256}$$

where $\ell = +, \times$; $e^{(+)}_{\alpha\beta}$ and $e^{(\times)}_{\alpha\beta}$ are bases of plus $(+)$ and cross $(\times)$ polarization states with $e^{(\ell)}_{\alpha\beta}(\mathbf{k}) e^{(\ell')\alpha\beta}(\mathbf{k}) = 2\delta_{\ell\ell'}$. Using

$$\hat{h}_\ell(\mathbf{x}, t) \equiv \frac{1}{2} \int \frac{d^3k}{(2\pi)^{3/2}} \hat{C}^{(t)}_{\alpha\beta}(\mathbf{x}, t; \mathbf{k}) e^{(\ell)\alpha\beta}(\mathbf{k}) = \int \frac{d^3k}{(2\pi)^{3/2}} \left[e^{i\mathbf{k}\cdot\mathbf{x}} \tilde{h}_{\ell\mathbf{k}}(t) \hat{a}_{\ell\mathbf{k}} + \text{h.c.} \right], \tag{257}$$

(255) becomes

$$\delta^2 S_g = \frac{1}{8\pi G} \int a^3 \sum_\ell \left(\dot{\hat{h}}^2_\ell - \frac{1}{a^2} \hat{h}_\ell^{\ ,\alpha} \hat{h}_{\ell,\alpha} \right) dt d^3x. \tag{258}$$

The equation of motion becomes ($v_g \equiv z_g \hat{h}_\ell$ and $z_g \equiv a$):

$$\ddot{\hat{h}}_\ell + 3H\dot{\hat{h}}_\ell - \frac{\Delta}{a^2} \hat{h}_\ell = \frac{1}{a^3} \left[v_g'' - \left(\frac{z_g''}{z_g} + \Delta \right) v_g \right] = 0. \tag{259}$$

Equal time commutation relation:

$$[\hat{h}_\ell(\mathbf{x},t),\dot{\hat{h}}_\ell(\mathbf{x}',t)] = 4\pi G\frac{i}{a^3}\delta^3(\mathbf{x}-\mathbf{x}'), \quad \delta\hat{\pi}_{h_\ell}(\mathbf{x},t)\equiv\partial\mathcal{L}/\partial\dot{\hat{h}}_\ell = \frac{1}{4\pi G}a^3\dot{\hat{h}}_\ell. \tag{260}$$

$$\tilde{h}_{\ell\mathbf{k}}(t)\dot{\tilde{h}}^*_{\ell\mathbf{k}}(t) - \tilde{h}^*_{\ell\mathbf{k}}(t)\dot{\tilde{h}}_{\ell\mathbf{k}}(t) = 4\pi G\frac{i}{a^3}. \tag{261}$$

For $z''_g/z_g = n_g/\eta^2$ with $n_g =$ constant (259) has an exact solution:

$$\tilde{h}_{\ell\mathbf{k}}(\eta) = \frac{\sqrt{\pi|\eta|}}{2a}\left[c_{\ell1}(\mathbf{k})H^{(1)}_{\nu_g}(k|\eta|) + c_{\ell2}(\mathbf{k})H^{(2)}_{\nu_g}(k|\eta|)\right]\sqrt{4\pi G}, \quad \nu_g\equiv\sqrt{n_g+\frac{1}{4}}, \tag{262}$$

$$|c_{\ell2}(\mathbf{k})|^2 - |c_{\ell1}(\mathbf{k})|^2 = 1. \tag{263}$$

Power spectrum:

$$\mathcal{P}_{\hat{C}^{(t)}_{\alpha\beta}}(\mathbf{k},t)\equiv\frac{k^3}{2\pi^2}\int\langle\hat{C}^{(t)}_{\alpha\beta}(\mathbf{x}+\mathbf{r},t)\hat{C}^{(t)\alpha\beta}(\mathbf{x},t)\rangle_{\rm vac}e^{-i\mathbf{k}\cdot\mathbf{r}}d^3r, \tag{264}$$

with $\hat{a}_{\ell\mathbf{k}}|{\rm vac}\rangle\equiv 0$ for all $\mathbf{k}$. We can show

$$\mathcal{P}_{\hat{C}^{(t)}_{\alpha\beta}}(\mathbf{k},t) = 2\sum_\ell\mathcal{P}_{\hat{h}_\ell}(\mathbf{k},t) = 2\sum_\ell\frac{k^3}{2\pi^2}\left|\tilde{h}_{\ell\mathbf{k}}(t)\right|^2. \tag{265}$$

Each $\hat{h}_\ell$ in Eq. (258) can be corresponded to a minimally coupled scalar field without potential with a normalization $\hat{h}_\ell = \sqrt{4\pi G}\hat{\phi}$. Assuming equal contributions from each polarization:

$$\mathcal{P}^{1/2}_{\hat{C}^{(t)}_{\alpha\beta}} = 2\mathcal{P}^{1/2}_{\hat{h}_\ell} = \sqrt{16\pi G}\mathcal{P}^{1/2}_{\hat{\phi}}. \tag{266}$$

Summary

- Taking suitable gauge condition is essential for proper handling.
- Gauge ready approach is practically convenient.
- Large scale evolutions: characterized by conserved quantities.
  - φ_v : conserved in the super-sound-horizon scale.
    From $C(\mathbf{x})$ follows every perturbation variable.
  - $C^{(t)}_{\alpha\beta}$: conserved in the super-horizon scale.
  - Rotation mode : angular momentum is conserved.
  - Conserved independently of changing equation of state, potential, and gravity theories.
    Assuming: near flat model, negligible stresses, and ignoring the transient solutions.
- Quantum fluctuations magnified by inflation mechanism provide plausible seeds for the large scale structures.
- Unified analyses of quantum generation and classical evolution in generalized gravity.
- These results are based on linear analyses.
  In linear theory, we have no 'structure formation', though!
- The original equations, both classical and quantum, are highly nonlinear.

4. CMB Anisotropies:

Large angular scale ($\theta > 1^o$):

Superhorizon scale at recombination.
Photon geodesic equation [37, 54] (Sachs-Wolfe 1967)

$$\tilde{k}^a{}_{;b}\tilde{k}^b = 0 = \tilde{k}^b\tilde{k}_b, \tag{267}$$

$$\frac{\tilde{T}_O}{\tilde{T}_E} = \frac{(\tilde{k}^a\tilde{u}_a)_O}{(\tilde{k}^b\tilde{u}_b)_E}. \tag{268}$$

Fluctuations reflect the initial conditions.
$\Rightarrow$ Window to the early universe (Inflation era).

Small angular scale ($\theta < 1^o$):

Subhorizon scale at recombination: local physical processes at recombination are important.
Thompson scattering couples radiation and dust.
Boltzmann equation [40]:

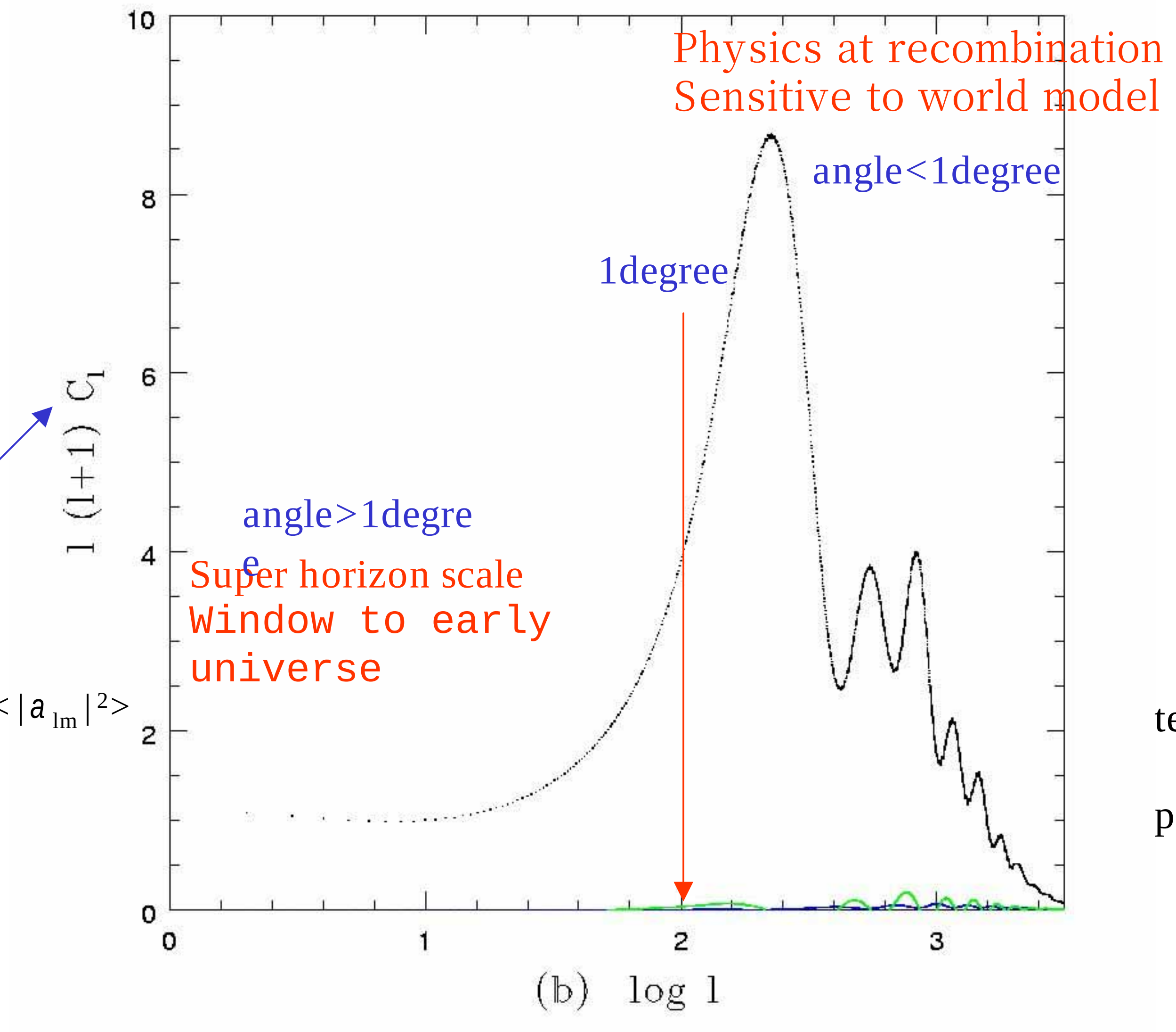
$$\frac{d\tilde{f}}{d\lambda} = p^a \frac{\partial \tilde{f}}{\partial x^a} - \tilde{\Gamma}^a_{bc} p^b p^c \frac{\partial \tilde{f}}{\partial p^a} = C[\tilde{f}],$$
$$\tilde{T}_{ab} = \int \frac{\sqrt{-\tilde{g}} d^3 p^{123}}{|p_0|} p_a p_b \tilde{f},$$
$$\tilde{f} = f + \delta f, \quad \frac{\delta T}{T} = \frac{1}{4} \frac{\int \delta f q^3 dq}{\int f q^3 dq}. \tag{269}$$

Polarizations (f_i, I, Q, U, V) are important as well.
Fluctuations are sensitive to local physics thus, sensitive to cosmological parameters [7, 6]:
Ω_b, Ω_γ, Ω_{CDM}, Ω_ν, Ω_{ν_m}, Ω_ϕ, H, K, Λ, ...

Gauge-ready formulation in generalized gravity [28]
Code based on:

- synchronous gauge (CMBFAST) = CDM comoving gauge (CAMB):
  CDM is pressureless, thus $v_{\rm CDM} = 0$ (comoving gauge)
  implies $\alpha = 0$ (synchronous gauge).
- uniform-curvature gauge
- uniform-expansion gauge
- zero-shear gauge (in trouble)

Theoretical predictions of temperature and polarization anisotropies



$C_l \equiv (2l+1)^{-1} \sum_m <|a_{lm}|^2>$

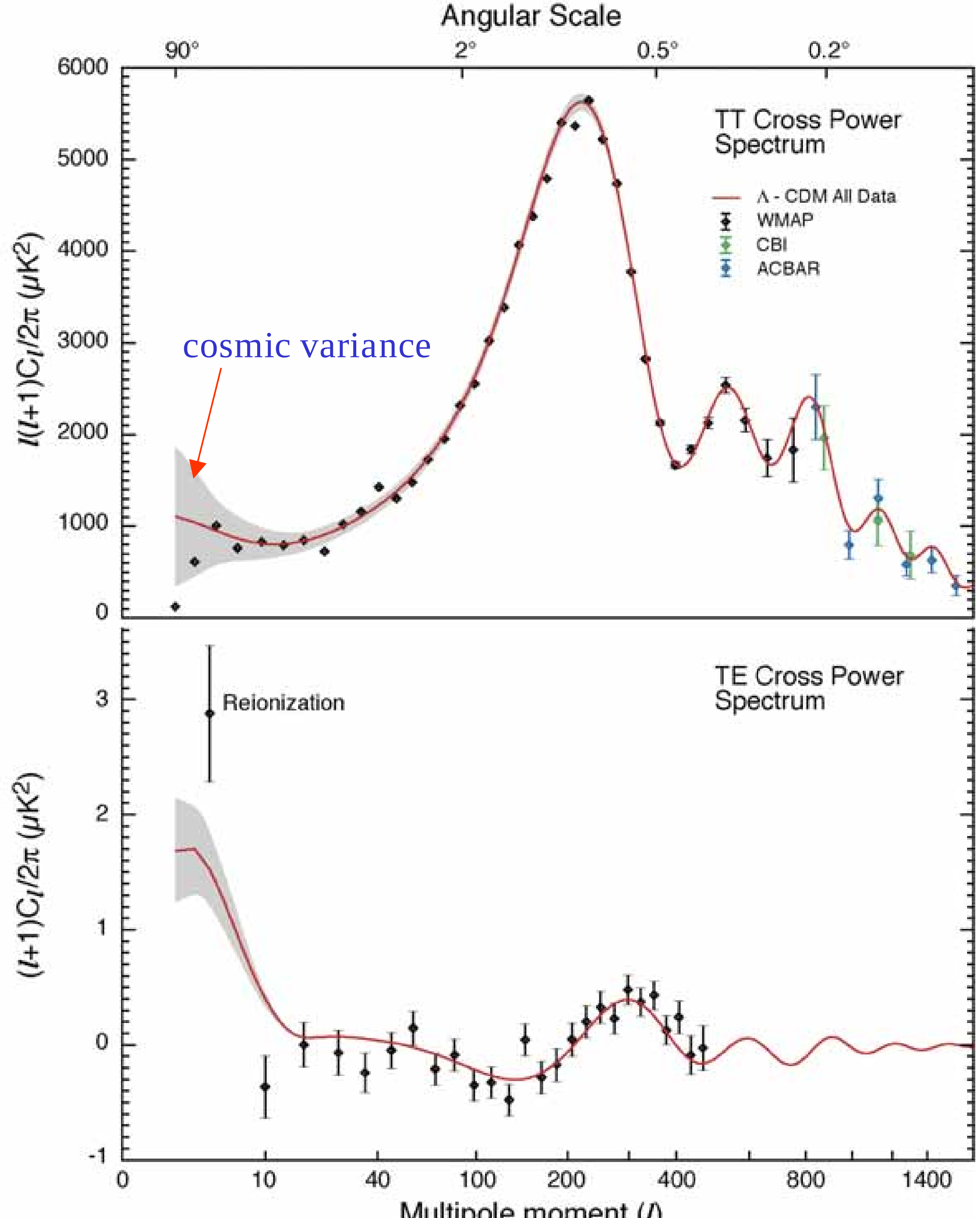
temperature ------

polarization ------

$\delta T / T \equiv \sum a_{lm} Y_{lm}$

WMAP

Temperature-polarization anisotropies



Bennett et al, ApJS **148**, 1 (2003)

Sachs-Wolfe effect:

Introduce the photon four-velocity (e^α and δe^α are based on $g^{(3)}_{\alpha\beta}$):

$$\tilde{k}^0 \equiv \frac{1}{a}(\nu + \delta\nu), \quad \tilde{k}^\alpha \equiv -\frac{\nu}{a}(e^\alpha + \delta e^\alpha);$$

$$\tilde{k}_0 = -a\nu\left(1 + \frac{\delta\nu}{\nu} + 2A - B_\alpha e^\alpha\right), \quad \tilde{k}_\alpha = -a\nu\left(e_\alpha + \delta e_\alpha + B_\alpha + 2C_{\alpha\beta}e^\beta\right). \tag{270}$$

We have

$$\frac{d}{d\lambda} = \frac{\partial x^a}{\partial\lambda}\frac{\partial}{\partial x^a} = \tilde{k}^a\partial_a = \frac{\nu}{a}\left(\partial_0 - e^\alpha\partial_\alpha + \frac{\delta\nu}{\nu}\partial_0 - \delta e^\alpha\partial_\alpha\right). \tag{271}$$

Thus,

$$\frac{d}{dy} \equiv \partial_0 - e^\alpha\partial_\alpha, \tag{272}$$

is a derivative along the background photon four-velocity.
The null and geodesic equations give:

$$\tilde{k}^a\tilde{k}_a = \nu^2\left[e^\alpha e_\alpha - 1 + 2\left(e^\alpha\delta e_\alpha - \frac{\delta\nu}{\nu} - A + B_\alpha e^\alpha + C_{\alpha\beta}e^\alpha e^\beta\right) = 0,\right. \tag{273}$$

$$\begin{aligned}\tilde{k}^0{}_{;b}\tilde{k}^b = \frac{\nu^2}{a^2}\Bigg[&\frac{(a\nu)'}{a\nu} + \left(\frac{\delta\nu}{\nu}\right)' + 2\frac{\nu'}{\nu}\frac{\delta\nu}{\nu} - \frac{\delta\nu_{,\alpha}}{\nu}e^\alpha + 2\frac{a'}{a}e^\alpha\delta e_\alpha + A' - 2\frac{a'}{a}A\\ &+ \left(B_{\alpha|\beta} + C'_{\alpha\beta} + 2\frac{a'}{a}C_{\alpha\beta}\right)e^\alpha e^\beta - 2\left(A_{,\alpha} - \frac{a'}{a}B_\alpha\right)e^\alpha\Bigg] = 0.\end{aligned} \tag{274}$$

To the background order:

$$e^\alpha e_\alpha = 1, \quad \nu \propto a^{-1}. \tag{275}$$

Using eqs. (272,273), eq. (274) becomes

$$\frac{d}{dy}\left(\frac{\delta\nu}{\nu} + A\right) = A_{,\alpha}e^\alpha - \left(B_{\alpha|\beta} + C'_{\alpha\beta}\right) e^\alpha e^\beta. \tag{276}$$

Thus

$$\left(\frac{\delta\nu}{\nu} + A\right)\Bigg|_E^O = \int_E^O \left[A_{,\alpha}e^\alpha - \left(B_{\alpha|\beta} + C'_{\alpha\beta}\right) e^\alpha e^\beta\right] dy, \tag{277}$$

where the integral is along the ray's null-geodesic path from E the emitted event at the intersection of the ray and the last scattering surface to O the observed event here and now.

The temperatures of the CMB at two different points (O and E) along a single null-geodesic ray in a given observational direction are [37, 54]

$$\frac{\tilde{T}_O}{\tilde{T}_E} \equiv \frac{1}{1+\tilde{z}} \equiv \frac{(\tilde{k}^a \tilde{u}_a)_O}{(\tilde{k}^b \tilde{u}_b)_E}, \tag{278}$$

where $\tilde{u}_a$ at O and E are the local four-velocities of the observer and the emitter, respectively.

Using eqs. (91,270) we have

$$\tilde{k}^a \tilde{u}_a = -\nu\left(1 + \frac{\delta\nu}{\nu} + A + \frac{1}{\mu+p}Q_\alpha e^\alpha\right). \tag{279}$$

Thus

$$\frac{\delta T}{T}\Bigg|_O = \frac{\delta T}{T}\Bigg|_E + \frac{1}{\mu+p}Q_\alpha e^\alpha\Big|_E^O + \int_E^O \left[A_{,\alpha}e^\alpha - \left(B_{\alpha|\beta} + C'_{\alpha\beta}\right) e^\alpha e^\beta\right] dy. \tag{280}$$

The most general expressions: [25]

$$\frac{\delta T}{T}\bigg|_O = \frac{\delta T}{T}\bigg|_E - v_{,\alpha}e^\alpha\bigg|_E^O + \int_E^O \left(-\varphi' + \alpha_{,\alpha}e^\alpha - \frac{1}{a}\chi_{,\alpha|\beta}e^\alpha e^\beta\right) dy$$
$$+ v_\alpha^{(v)}e^\alpha\bigg|_E^O - \int_E^O \Psi_{\alpha|\beta}^{(v)}e^\alpha e^\beta dy - \int_E^O C_{\alpha\beta}^{(t)\prime}e^\alpha e^\beta dy. \quad (281)$$

$\delta T|_O$ is gauge independent because it is a difference between different directions.

For the scalar-type we have:

Integrated Sachs-Wolfe

$$\frac{\delta T}{T}\bigg|_O = \frac{\delta T_\chi}{T}\bigg|_E - v_{\chi,\alpha}e^\alpha\bigg|_E^O + \alpha_\chi\bigg|_E + \int_E^O (\alpha_\chi - \varphi_\chi)' dy. \quad (282)$$

In matter dominated era with $K = 0 = \Lambda$, in the large angular scale:

$$\frac{\delta T}{T}\bigg|_O = -\frac{1}{3}\varphi_\chi\bigg|_E. \quad (283)$$

It is a straight relativistic result [34].

Angular anisotropies:

$$\frac{\delta T}{T}(\mathbf{e};\mathbf{x}_R) = \sum_{lm} a_{lm}(\mathbf{x}_R)Y_{lm}(\mathbf{e}), \quad \langle a_l^2\rangle \equiv \langle |a_{lm}(\mathbf{x}_R)|^2\rangle_{\mathbf{x}_R}. \quad (284)$$

For $K = 0 = \Lambda$, in matter dominated era [1]:

$$\langle a_l^2\rangle_S = \frac{4\pi}{25}\int_0^\infty \mathcal{P}_{\varphi_v}(k)j_l^2(kx)d\ln k, \quad x \equiv \frac{2}{H_0}, \quad (285)$$

$$\langle a_l^2\rangle_T = \frac{9\pi^3}{4}\frac{\Gamma(l+3)}{\Gamma(l-1)}\int_0^\infty \mathcal{P}_{C_{\alpha\beta}^{(t)}}(k)\left|\frac{2}{\pi}\int_{\eta_e}^{\eta_o}\frac{j_2(k\eta)}{k\eta}\frac{j_l(k\eta_0 - k\eta)}{(k\eta_0 - k\eta)^2}kd\eta\right|^2 d\ln k. \quad (286)$$

5. Multi-component Case

Fluids:

(169,170) give:

$$\frac{H^2 c_s^2}{(\mu+p)a^3}\left[\frac{(\mu+p)a^3}{H^2 c_s^2}\dot{\varphi}_v\right]^{\cdot} - c_s^2\frac{\Delta}{a^2}\varphi_v = -\frac{H^2 c_s^2}{(\mu+p)a^3}\left[\frac{a^3}{c_s^2 H}\left(e+\frac{2}{3}\frac{\Delta}{a^2}\Pi\right)\right]^{\cdot} - 2H^2\frac{c_s^2}{\mu+p}\frac{\Delta}{a^2}\Pi, \qquad (287)$$

where (ignoring direct interactions among components in the background) [36, 13, 26]

$$e \equiv \delta p - c_s^2\delta\mu = \sum_k \delta p_{(k)} - \frac{\sum_l \dot{p}_{(l)}}{\sum_m \dot{\mu}_{(m)}}\sum_k \delta\mu_{(k)} = \frac{1}{2}\sum_{k,l}\frac{(\mu_{(k)}+p_{(k)})(\mu_{(l)}+p_{(l)})}{\mu+p}(c_{(k)}^2 - c_{(l)}^2)S_{(kl)} + \sum_k e_{(k)},$$

$$S_{(ij)} \equiv \frac{\delta\mu_{(i)}}{\mu_{(i)}+p_{(i)}} - \frac{\delta\mu_{(j)}}{\mu_{(j)}+p_{(j)}}. \qquad (288)$$

- φ_v is conserved in the large-scale limit when:

(*1*) single ideal fluid ($e = 0 = \pi^{(s)}$)
(*2*) multiple pressureless ideal fluids $c_{(i)}^2 = 0$
(*3*) multiple ideal fluids with $c_{(i)}^2 = c_{(j)}^2$
(*4*) multiple ideal fluids with adiabatic perturbation, $S_{(ij)} = 0$

Curvature(adiabatic) and isocurvature:

Assuming ideal fluids, and ignoring direct interactions among fluids (123,124) give [13, 26]:

$$\frac{\mu + p}{a^2 H \mu}\left[\frac{H^2}{a(\mu+p)}\left(\frac{a^3\mu}{H}\delta_v\right)^{\cdot}\right]^{\cdot} + c_s^2\frac{k^2}{a^2}\delta_v = -\frac{k^2 - 3K}{a^2}\frac{e}{\mu}, \tag{289}$$

$$\ddot{S}_{(ij)} + \left[2 - \frac{3}{2}(c_{(i)}^2 + c_{(j)}^2)\right] H\dot{S}_{(ij)} - \frac{3}{2}H(c_{(i)}^2 - c_{(j)}^2)\sum_k \frac{\mu_{(k)} + p_{(k)}}{\mu + p}\left(\dot{S}_{(ik)} + \dot{S}_{(jk)}\right)$$
$$+\frac{1}{2}\frac{k^2}{a^2}\left[(c_{(i)}^2 + c_{(j)}^2)S_{(ij)} + (c_{(i)}^2 - c_{(j)}^2)\sum_k \frac{\mu_{(k)} + p_{(k)}}{\mu + p}\left(S_{(ik)} + S_{(jk)}\right)\right]$$
$$= -\frac{k^2}{a^2}(c_{(i)}^2 - c_{(j)}^2)\frac{\delta\mu_v}{\mu + p}. \tag{290}$$

Fields:

In the uniform-curvature gauge (129) gives [56]:

$$\delta\ddot{\phi}^{(i)}_{\varphi} + 3H\delta\dot{\phi}^{(i)}_{\varphi} + \frac{k^2}{a^2}\delta\phi^{(i)}_{\varphi} = -\sum_k \left[V^{,(i)}_{\ \ ,(k)} - \frac{8\pi G}{a^3}\left(\frac{a^3}{H}\dot{\phi}^{(i)}\dot{\phi}_{(k)}\right)^{\cdot}\right]\delta\phi^{(k)}_{\varphi}. \tag{291}$$

Curvature and isocurvature: [13, 26]

$$\frac{H^2}{(\mu+p)a^3}\left[\frac{(\mu+p)a^3}{H^2}\dot{\varphi}_v\right]^{\cdot} + \frac{k^2}{a^2}\varphi_v = \frac{2H^2}{(\mu+p)a^3}\left[\frac{a^3}{(\mu+p)H}\sum_{k,l}V_{,(k)}\dot{\phi}_{(k)}\dot{\phi}^2_{(l)}\delta\phi_{(kl)}\right]^{\cdot}, \tag{292}$$

$$\begin{aligned} &\delta\ddot{\phi}_{(ij)} + \left(-3\dot{H} + \frac{k^2}{a^2}\right)\delta\phi_{(ij)} - \sum_k\left(V_{,(i)(k)}\frac{\dot{\phi}_{(k)}}{\dot{\phi}_{(i)}}\delta\phi_{(ik)} - V_{,(j)(k)}\frac{\dot{\phi}_{(k)}}{\dot{\phi}_{(j)}}\delta\phi_{(jk)}\right) \\ &\quad -\frac{3}{2}H\left[(c^2_{(i)} + c^2_{(j)})\delta\dot{\phi}_{(ij)} + (c^2_{(i)} - c^2_{(j)})\sum_k\frac{\mu_{(k)}+p_{(k)}}{\mu+p}\left(\delta\dot{\phi}_{(ik)} + \delta\dot{\phi}_{(jk)}\right)\right] \\ &\quad = 3H(c^2_{(i)} - c^2_{(j)})\frac{\delta\mu_v}{\mu+p}, \end{aligned} \tag{293}$$

where

$$\delta\phi_{(ij)} \equiv \frac{\delta\phi_{(i)}}{\dot{\phi}_{(i)}} - \frac{\delta\phi_{(j)}}{\dot{\phi}_{(j)}}, \quad \delta\mu_v = -\left(\frac{\mu+p}{H}\dot{\varphi}_v - \frac{2}{\mu+p}\sum_{k,l}V_{,(k)}\dot{\phi}_{(k)}\dot{\phi}^2_{(l)}\delta\phi_{(kl)}\right). \tag{294}$$

- φ_v is conserved in the large-scale when:

(1) single component, *(2)* $\delta\phi_{(ij)} = 0$, *(3)* $V =$ constant,

(4) $V_{,(i)} \propto \dot{\phi}_{(i)}$ $(\supset 3H\dot{\phi}_{(i)} + V_{,(i)} = 0$, slow-roll).

Lecture 4

1. **Cosmological perturbations: Summary**
2. **Generalized gravity theories**
3. **Structure of the theory**
4. **Second-order perturbations**
5. **Zero-pressure fluid**
   5.1 **Second-order perturbations: Relativistic-Newtonian correspondence**
   5.2 **Third-order perturbations: Pure relativistic corrections**

"the linear perturbations are so surprisingly simple that a perturbation analysis accurate to second order may be feasible ... One could then judge the domain of validity of the linear treatment and, more important, gain some insight into the non-linear effects."

Sachs and Wolfe (1967)

1. Cosmological Perturbations: Summary

Methods:

- **Relativistic:**
  1. Einstein equations (Lifshitz 1946)
  2. Covariant equations ($1+3$, $\tilde{u}_a$; Hawking 1966)
  3. ADM equations ($3+1$, $\tilde{n}_a$; Bardeen 1980)
  4. Action formulation (Lukash 1980; Mukhanov 1988)
- **Newtonian:**
  1. Hydrodynamic equations (Bonner 1957)

Relativistic-Newtonian **correspondence** in the zero-pressure case.

★ **True** even to the second order!

Three perturbation types:

1. Scalar-type: density fluctuations
2. Vector-type: rotation
3. Tensor-type: gravitational wave

To linear-order, **decouple** in Friedmann background

★ **Couple** to the second order!

Classical Evolution:

1. Scalar-type: super-sound-horizon scale conservation
2. Rotation: angular momentum conservation
3. Gravitational wave: super-horizon scale conservation

★ **True** even to the second order!

Correspondences: [24, 17]

$$\delta\mu_v,\ \varphi_\chi,\ v_\chi\ (\kappa_v) \sim \text{Newtonian } \delta\varrho,\ \delta\Phi,\ \mathbf{u}$$

even to the second order

$$\delta\phi_\varphi \sim \phi \text{ in QFCS}$$

$$\varphi_v = \varphi_{\delta\phi} \sim \text{super-sound-horizon conservation}$$

$\delta\mu_v \equiv \delta_v - \dot{\mu} a v =$ perturbed density $\delta\mu$ in the comoving gauge ($v \equiv 0$).
$\varphi_\chi \equiv \varphi - H\chi =$ perturbed three-space curvature φ in the zero-shear gauge ($\chi \equiv 0$).
$v_\chi \equiv v - \frac{1}{a}\chi =$ perturbed velocity v in the zero-shear gauge ($\chi \equiv 0$).
$\kappa_v \equiv \kappa + (3\dot{H} + \frac{\Delta}{a^2})av =$ perturbed expansion scalar κ in the comoving gauge ($v \equiv 0$).
$\varphi_v \equiv \varphi - aHv =$ perturbed three-space curvature φ in the comoving gauge ($v \equiv 0$).
$\varphi_{\delta\phi} \equiv \varphi - \frac{H}{\dot{\phi}}\delta\phi =$ perturbed three-space curvature φ in the uniform-field gauge ($\delta\phi \equiv 0$).
$\delta\phi_\varphi \equiv \delta\phi - \frac{\dot{\phi}}{H}\varphi =$ perturbed scalar field $\delta\phi$ in the uniform-curvature gauge ($\varphi \equiv 0$).

These are all gauge invariant.

Perturbed action: (Lukash 1980; Mukhanov 1988)

$$\delta^2 S = \frac{1}{2}\int a^3 Q \left(\dot{\Phi}^2 - c_A^2 \frac{1}{a^2}\Phi^{,\alpha}\Phi_{,\alpha} \right) dt d^3 x.$$

where

$$\begin{cases} \Phi = \varphi_v & Q = \frac{\mu+p}{c_s^2 H^2} & c_A^2 \to c_s^2 & \text{(fluid)} \\ \Phi = \varphi_{\delta\phi} & Q = \frac{\dot{\phi}^2}{H^2} & c_A^2 \to 1 & \text{(field)} \\ \Phi = C^{(t)}_{\alpha\beta} & Q = \frac{1}{8\pi G} & c_A^2 \to 1 & \text{(GW)} \end{cases}$$

$\varphi_v \equiv \varphi - aHv$ and $\varphi_{\delta\phi} \equiv \varphi - \frac{H}{\dot{\phi}}\delta\phi$: gauge-invariant combinations.

★ Generalized gravity theories as well!

Equation of motion (Field-Shepley 1968) $v \equiv z\Phi$ and $z \equiv a\sqrt{Q}$:

$$\frac{1}{a^3 Q}\left(a^3 Q \dot{\Phi}\right)^{\cdot} - c_A^2 \frac{\Delta}{a^2}\Phi = \frac{1}{a^2 z}\left[v'' - \left(\frac{z''}{z} + c_A^2 \Delta \right) v \right] = 0.$$

Large-scale solution:

$$\Phi(\mathbf{x}, t) = C(\mathbf{x}) - D(\mathbf{x}) \int_0^t \frac{dt}{a^3 Q}.$$

2. Generalized $f(\phi, R)$ gravity:

Introduce [12]:

$$\tilde{L} = \frac{1}{2} f(\tilde{\phi}, \tilde{R}) - \frac{1}{2}\omega(\tilde{\phi})\tilde{\phi}^{\cdot a}\tilde{\phi}_{,a} - V(\tilde{\phi}) + \tilde{L}_m. \tag{295}$$

Special cases: $F \equiv \frac{\partial f}{\partial R}$, ignoring tildes

Minimally coupled scalar field	$L = \frac{1}{2\kappa^2}R - \frac{1}{2}\phi^{\cdot a}\phi_{,a} - V(\phi)$
Nonminimally coupled scalar field	$L = \frac{1}{2}\left(\kappa^{-2} - \xi\phi^2\right)R - \frac{1}{2}\phi^{\cdot a}\phi_{,a} - V(\phi)$
Brans-Dicke theory	$L = \phi R - \omega\frac{\phi^{\cdot a}\phi_{,a}}{\phi}$
Generalizes scalar-tensor theory	$L = \phi R - \omega(\phi)\frac{\phi^{\cdot a}\phi_{,a}}{\phi} - V(\phi)$
Induced gravity	$L = \frac{1}{2}\epsilon\phi^2 R - \frac{1}{2}\phi^{\cdot a}\phi_{,a} - \frac{1}{4}\lambda(\phi^2 - v^2)^2$
R^2 gravity	$L = \frac{1}{2}\left(R - \frac{R^2}{6M^2}\right)$
$F(\phi)R$ gravity	$L = \frac{1}{2}F(\phi)R - \frac{1}{2}\omega(\phi)\phi^{\cdot a}\phi_{,a} - V(\phi)$
$f(R)$ gravity	$L = \frac{1}{2}f(R)$
Low-energy string theory	$L = \frac{1}{2}e^{-\phi}\left(R + \phi^{\cdot a}\phi_{,a}\right)$

Conformally equivalent to Einstein's theory [12, 22, 28].

Unified Analyses in Generalized $f(\phi, R)$ gravity: [31]

$$\tilde{S} = \int d^4x \sqrt{-\tilde{g}} \left[\frac{1}{2} f(\tilde{\phi}, \tilde{R}) - \frac{1}{2}\omega(\tilde{\phi})\tilde{\phi}^{;a}\tilde{\phi}_{,a} - V(\tilde{\phi}) \right].$$

Action $\quad \delta^2 S = \frac{1}{2}\int a^3 Q \left(\dot{\Phi}^2 - \frac{1}{a^2}\Phi^{,\alpha}\Phi_{,\alpha} \right) dt d^3x$

Scalar-type: $\quad \Phi = \varphi_{\delta\phi}, \quad Q = \frac{\omega\dot{\phi}^2 + 3\dot{F}^2/2F}{\left(H + \dot{F}/2F\right)^2}$

Tensor-type: $\quad \Phi = C^{(t)\alpha}_{\beta}, \quad Q = F$

Equation $\quad \frac{1}{a^3 Q}(a^3 Q\dot{\Phi})^{\cdot} - \frac{1}{a^2}\Delta\Phi = 0$

Large scale $\quad \Phi = C(\mathbf{x}) - D(\mathbf{x}) \int_0^t (a^3 Q)^{-1} dt$

Quantization $\quad [\hat{\Phi}(\mathbf{x}, t), \dot{\hat{\Phi}}(\mathbf{x}', t)] = \frac{i}{a^3 Q}\delta^3(\mathbf{x} - \mathbf{x}')$

Mode func. $\quad$ For $a\sqrt{Q} \propto \eta^q$ (include many inflation models)

$$\Phi_k(\eta) = \frac{\sqrt{\pi|\eta|}}{2a\sqrt{Q}} \left[c_1(k) H_\nu^{(1)}(k|\eta|) + c_2(k) H_\nu^{(2)}(k|\eta|) \right]$$

where $\nu \equiv \frac{1}{2} - q, \quad |c_2(k)|^2 - |c_1(k)|^2 = 1$

- In super-horizon scale, ignoring transient one, $\Phi(\mathbf{x}, t) = C(\mathbf{x})$.
- Conserved independently of changing gravity theory.
- Unified analysis allows us to handle transitions among gravity theories.

More generalized Gravity Theories

1. Generalized $f(\phi, R)$ gravity: [20, 22, 31]

$$\tilde{S} = \int \left[\frac{1}{2} f(\tilde{\phi}, \tilde{R}) - \frac{1}{2}\omega(\tilde{\phi})\tilde{\phi}^{;c}\tilde{\phi}_{,c} - V(\tilde{\phi}) + \tilde{L}_{(c)} \right] \sqrt{-\tilde{g}} d^4x. \tag{296}$$

2. Tachyonic generalization: [30] $\tilde{X} \equiv \frac{1}{2}\tilde{\phi}^{;c}\tilde{\phi}_{,c}$

$$\tilde{S} = \int \left[\frac{1}{2} f(\tilde{\phi}, \tilde{R}, \tilde{X}) + \tilde{L}_{(c)} \right] \sqrt{-\tilde{g}} d^4x.$$

3. String corrections: [27]

$$\tilde{L}_{(c)} = \xi(\tilde{\phi}) \Big[c_1 \left(\tilde{R}^{abcd}\tilde{R}_{abcd} - 4\tilde{R}^{ab}\tilde{R}_{ab} + \tilde{R}^2 \right) + c_2 \tilde{G}^{ab}\tilde{\phi}_{,a}\tilde{\phi}_{,b} + c_3 \tilde{\phi}^{;a}{}_{a}\tilde{\phi}^{;b}\tilde{\phi}_{,b} + c_4 (\tilde{\phi}^{;a}\tilde{\phi}_{,a})^2 \Big]. \tag{297}$$

4. String axion coupling: [27]

$$\tilde{L}_{(c)} = \frac{1}{8}\nu(\tilde{\phi})\tilde{\eta}^{abcd}\tilde{R}_{ab}{}^{ef}\tilde{R}_{cdef}. \tag{298}$$

We can always derive a unified form: [31]

$$\delta^2 S = \frac{1}{2}\int a^3 Q \left(\dot{\Phi}^2 - c_A^2 \frac{1}{a^2}\Phi^{;\alpha}\Phi_{,\alpha} \right) dt d^3x. \tag{299}$$

★ Perhaps **"surprisingly simple"** indeed!

3. Structure of the Theory

Components: baryon (b), photon (γ), massless neutrinos (ν), massive neutrinos (ν_m), CDM (c), K, Λ, fields (ϕ, quintessence, dilaton, ...), ...

$$\tilde{G}_{ab} = 8\pi G \sum_\ell \tilde{T}^{(\ell)}_{ab}, \qquad \ell = b, \gamma, \nu, c, \nu_m, \phi, \dots$$

$$\Uparrow$$

$$\tilde{T}^{(\ell)}_{ab} = \begin{cases} (\tilde{\mu} + \tilde{p})\tilde{u}_a\tilde{u}_b + \tilde{p}\tilde{g}_{ab} + \tilde{\pi}_{ab} & \text{:fluid } (b, c, \Lambda) \\ \int \frac{\sqrt{-\tilde{g}}d^3p^{123}}{|p_0|} p_a p_b \tilde{f} & \text{:kinetic } (\gamma, \nu, \nu_m) \\ \tilde{\phi}_{,a}\tilde{\phi}_{,b} - (\frac{1}{2}\tilde{\phi}^{,c}\tilde{\phi}_{,c} + \tilde{V})\tilde{g}_{ab} & \text{:field } (\phi) \to \text{or GGT} \end{cases}$$

$$\left[\begin{array}{lll} \text{fluid:} & \tilde{T}^{(\ell);b}_{ab} = 0 & \to \text{energy, momentum conservation} \\ \text{kinetic:} & \frac{d}{d\lambda}\tilde{f} = C[\tilde{f}] & \to \text{hierarchy} \\ \text{field:} & \tilde{\phi}^{;c}{}_{c} - \tilde{V}_{,\tilde{\phi}} = 0 & \to \text{or GGT} \end{array} \right.$$

Fluids and kinetic components feel the generalized nature of gravity only through metric perturbations and background evolution.

Summary

- Taking suitable gauge condition is essential for proper handling.
- Gauge ready approach is practically convenient.
- Large scale evolutions: characterized by conserved quantities.
  - φ_v : conserved in the super-sound-horizon scale.
    From $C(\mathbf{x})$ follows every perturbation variable.
  - $C_{\alpha\beta}$: conserved in the super-horizon scale.
  - Rotation mode : angular momentum is conserved.
  - Conserved independently of changing equation of state, potential, and gravity theories.
    Assuming: near flat model, negligible stresses, and ignoring the transient solutions.
- Quantum fluctuations magnified by inflation mechanism provide plausible seeds for the large scale structures.
- Unified analyses of quantum generation and classical evolution in generalized gravity.
- These results are based on linear analyses.
  In linear theory, we have no 'structure formation', though!
- The original equations, both classical and quantum, are highly nonlinear.

Why linear theory?:

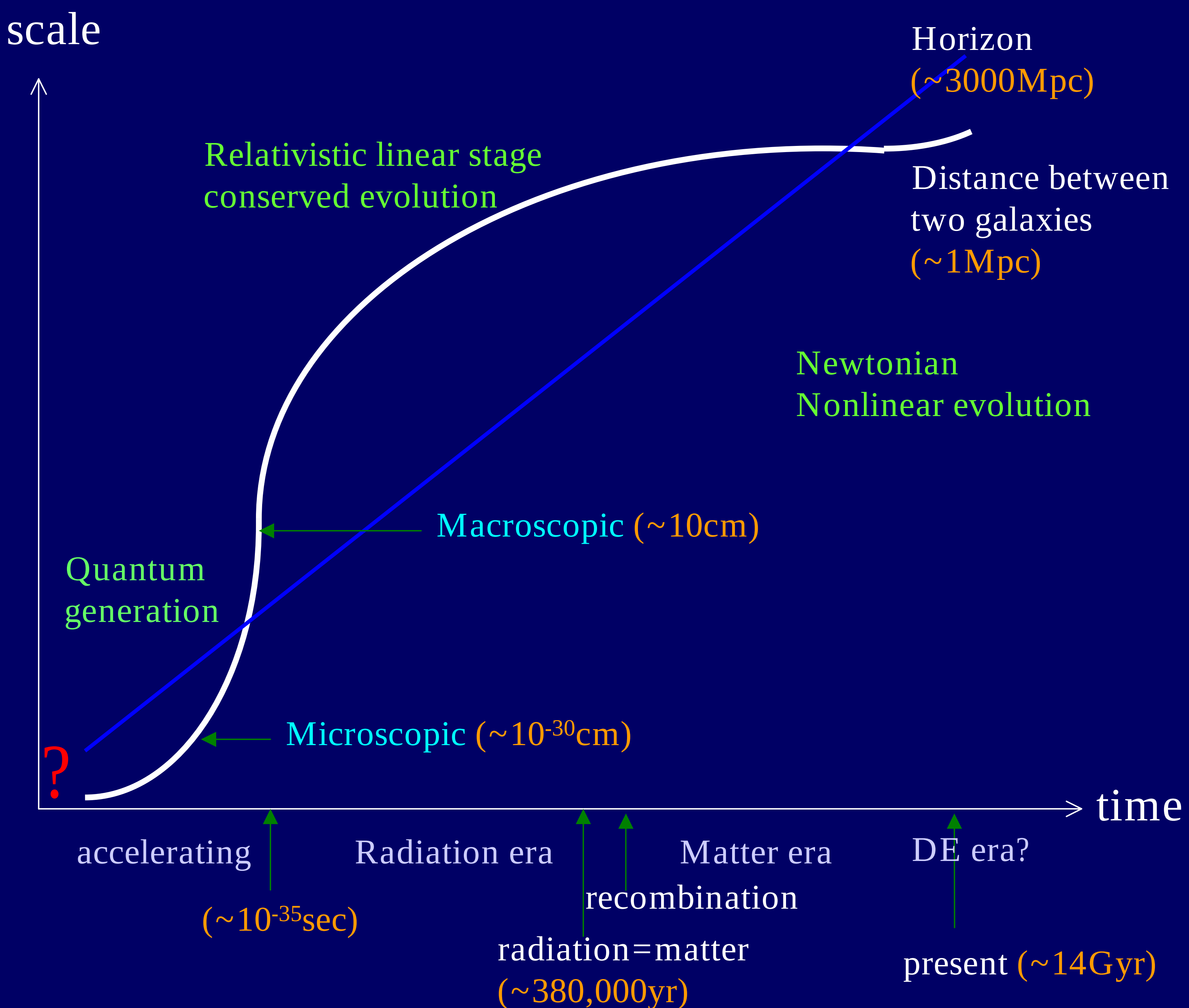
1. The CMB temperature and polarization anisotropies are very small $\frac{\delta T}{T} \sim 10^{-5}$.
2. The large-scale clustering of galaxies are approximately linear as the scale becomes large. Our own homogeneous and isotropic background world model relies on this assumption. Observations are not inconsistent with the assumption.

If the fluctuation is on $\sim 10^{-5}$ level, Taylor's series theorem guarantees the non-linear terms are negligible $\sim 10^{-10}$.

Still, considering that the basic equations are fully nonlinear, it is matter of whether we can ignore (or tolerate) the level of nonlinearities.

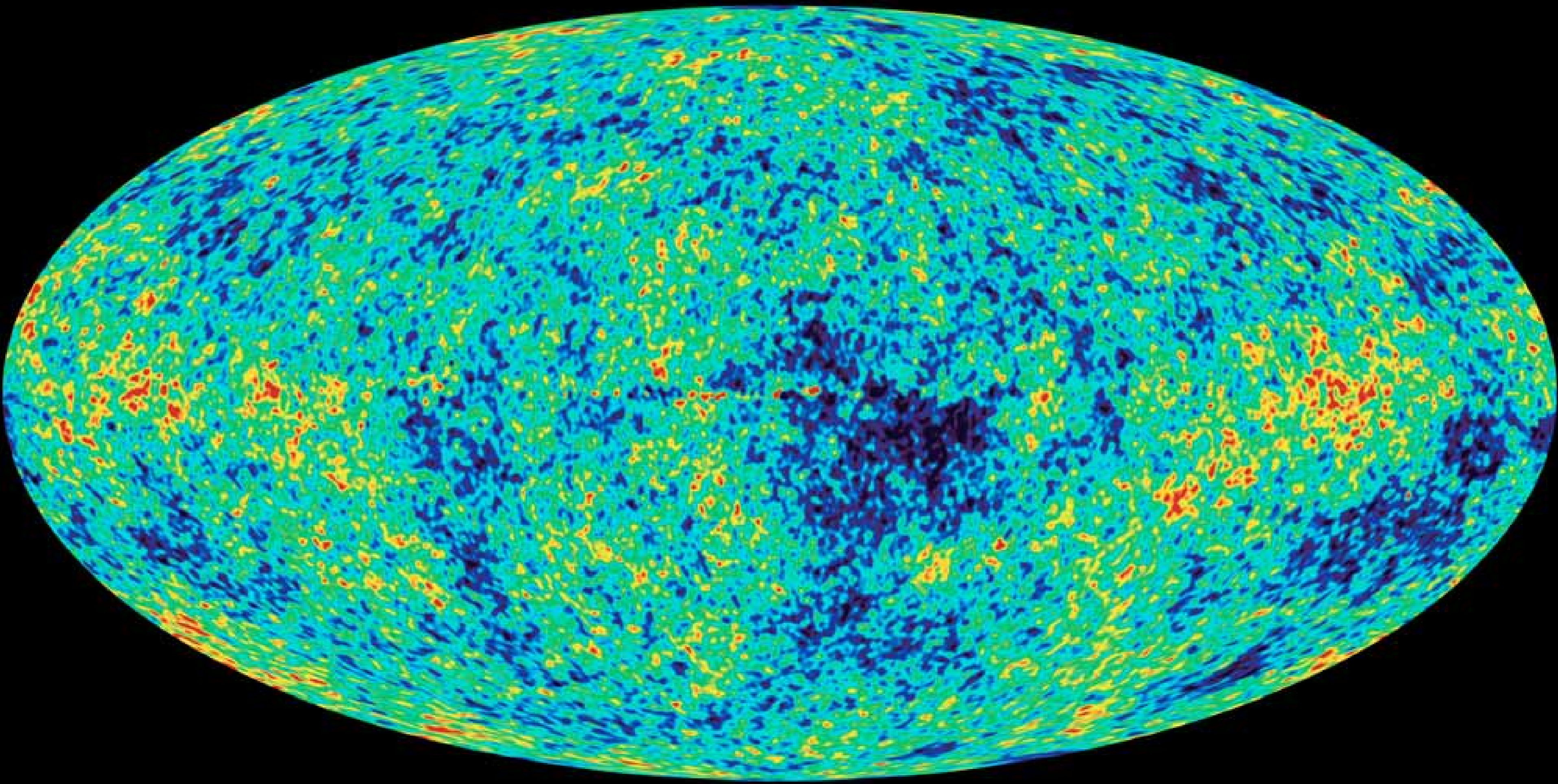
It looks we may currently **assume** linearity in the early universe and in the large-scale in the present era.

If the situation is linear, then we can handle both physics and mathematics very reliably.

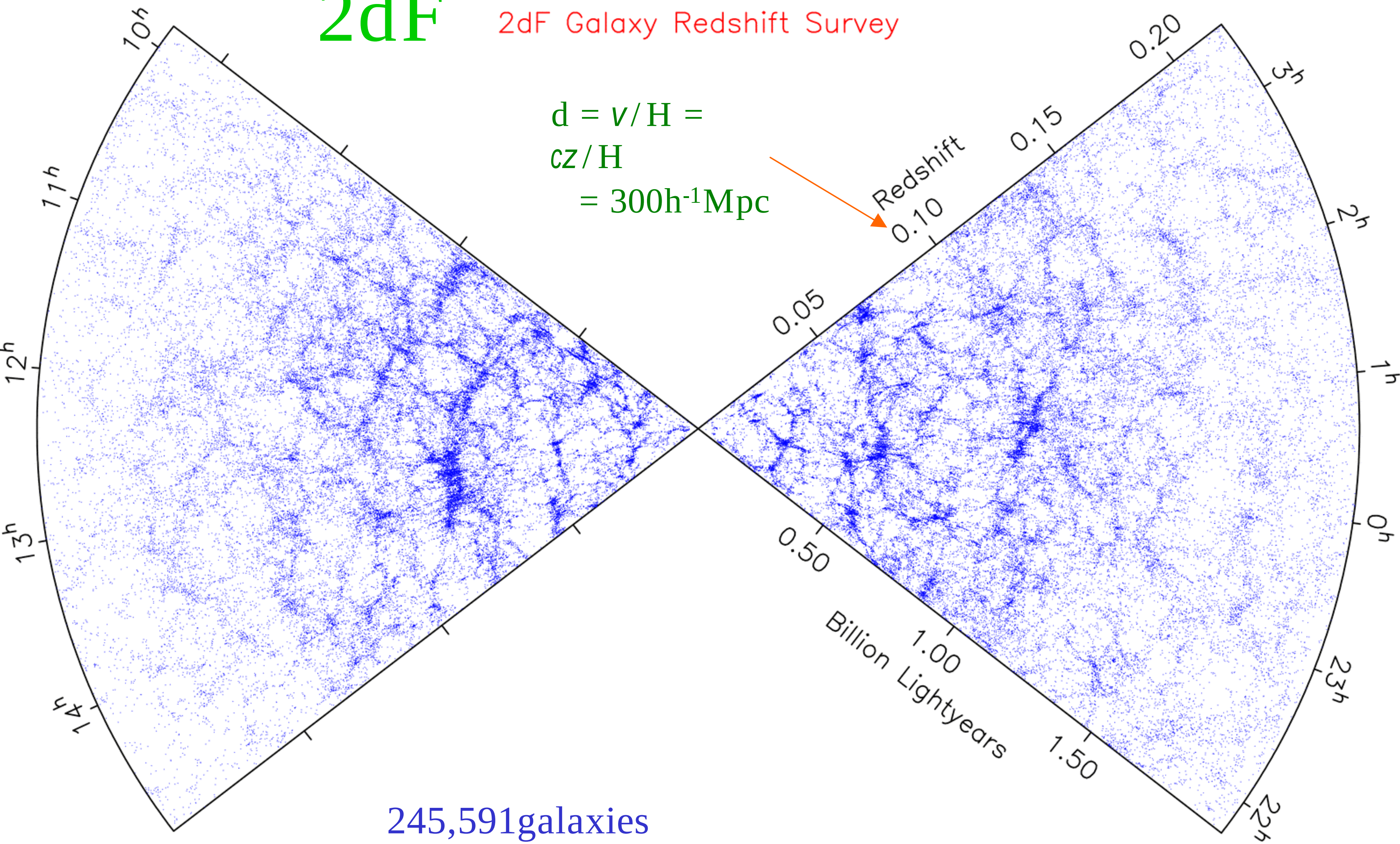


scale
Horizon
(~3000Mpc)
Relativistic linear stage
conserved evolution
Distance between
two galaxies
(~1Mpc)
Newtonian
Nonlinear evolution
Macroscopic (~10cm)
Quantum
generation
Microscopic (~10-30cm)
?
time
accelerating
Radiation era
Matter era
DE era?
recombination
(~10-35sec)
radiation=matter
(~380,000yr)
present (~14Gyr)

WMAP



2dF

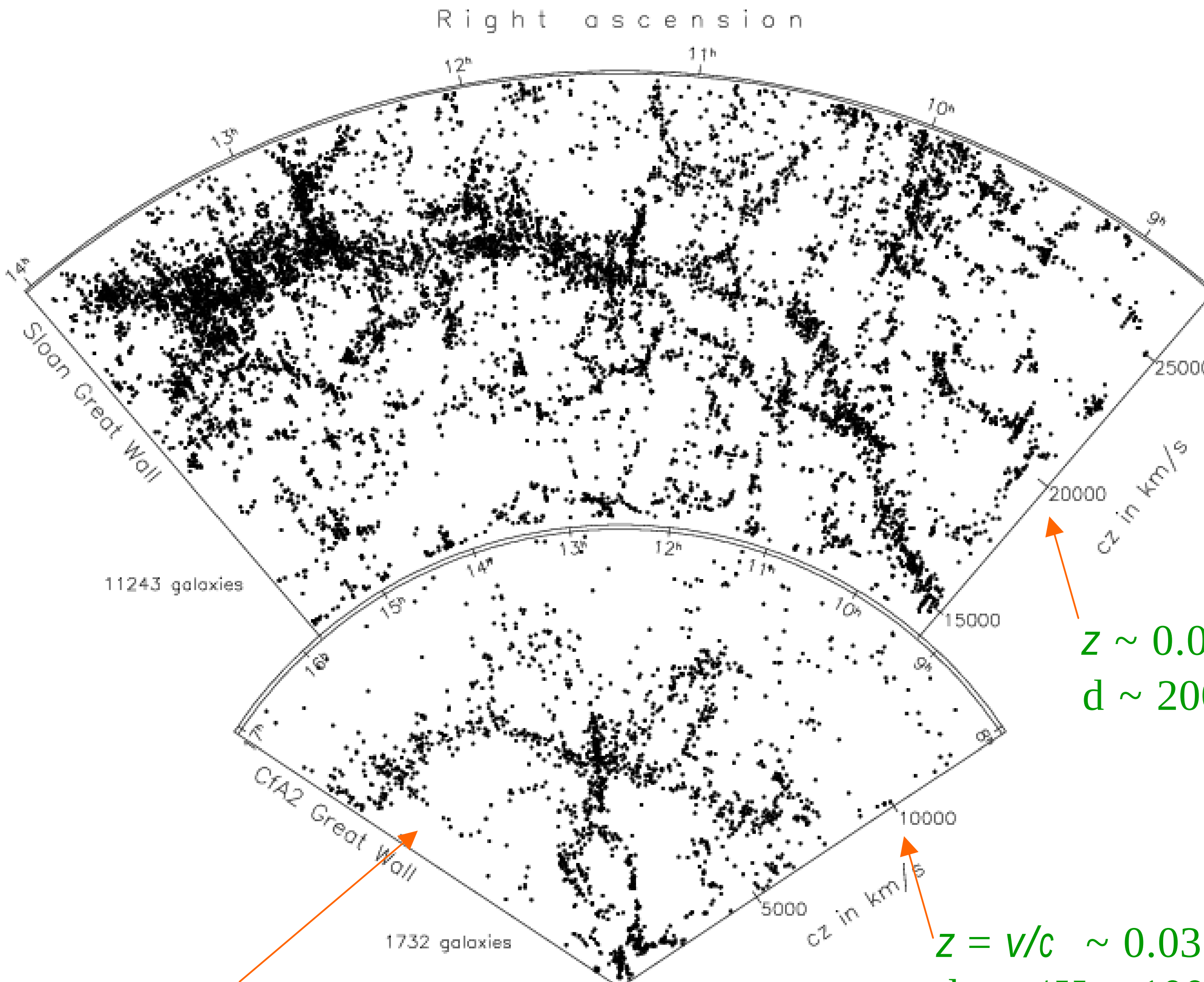


$d = v/H = cz/H = 300h^{-1}\mathrm{Mpc}$

245,591galaxies

Limiting magnitude : 19.45

SDSS



$z \sim 0.06$

$d \sim 200h^{-1}$Mpc

$z = v/c \sim 0.03$

$d = v/H \sim 100h^{-1}$Mpc

de Lapparent *etal.* (1986)

4. Second-order Perturbation [51]

The metric:

$$\tilde{g}_{00} \equiv -a^2(1+2A), \quad \tilde{g}_{0\alpha} \equiv -a^2 B_\alpha, \quad \tilde{g}_{\alpha\beta} \equiv a^2\left(g^{(3)}_{\alpha\beta} + 2C_{\alpha\beta}\right). \tag{300}$$

To the second-order:

$$A \equiv A^{(1)} + A^{(2)}, \quad B_\alpha \equiv B^{(1)}_\alpha + B^{(2)}_\alpha, \quad C_{\alpha\beta} \equiv C^{(1)}_{\alpha\beta} + C^{(2)}_{\alpha\beta}. \tag{301}$$

Inverse metric:

$$\begin{aligned} \tilde{g}^{00} &= \frac{1}{a^2}\left(-1 + 2A - 4A^2 + B_\alpha B^\alpha\right), \\ \tilde{g}^{0\alpha} &= \frac{1}{a^2}\left(-B^\alpha + 2AB^\alpha + 2B_\beta C^{\alpha\beta}\right), \\ \tilde{g}^{\alpha\beta} &= \frac{1}{a^2}\left(g^{(3)\alpha\beta} - 2C^{\alpha\beta} - B^\alpha B^\beta + 4C^\alpha_\gamma C^{\beta\gamma}\right). \end{aligned} \tag{302}$$

Decomposition:

$$\begin{aligned} A &\equiv \alpha, \\ B_\alpha &\equiv \beta_{,\alpha} + B^{(v)}_\alpha, \\ C_{\alpha\beta} &\equiv \varphi g^{(3)}_{\alpha\beta} + \gamma_{,\alpha|\beta} + C^{(v)}_{(\alpha|\beta)} + C^{(t)}_{\alpha\beta}, \end{aligned} \tag{303}$$

Similarly for the energy-momentum tensor.

“PRD **69** 104011 (2004)” [51]

1. Complete set of second-order perturbation equations in the gauge-ready form.
2. The most general Friedmann background with K and Λ.
3. (1) The most general imperfect fluid situation which includes multiple imperfect fluids with general interactions among them.

   (2) Minimally coupled scalar fields.

   (3) A class of generalized gravity theories.

   (4) Electromagnetic fields.

   (5) Null geodesic equation.

   (6) The relativistic Boltzmann equation.
4. Decomposed forms into three different modes with couplings appearing in the second-order.
5. Gauge issue resolved to the second (and all) order:

   gauge-invariant combinations, and gauge-ready formulation.
6. Derived the large-scale (super-sound-horizon) conserved quantity to the second-order: φ_v and $C^{(t)}_{\alpha\beta}$. These were first known by Salopek and Bond (1990).
7. Up to the second order in perturbations the relativistic fluid without pressure coincides exactly with the Newtonian one.

Large-scale conservation [55, 51]

Scalar-type: in super-sound-horizon scale

$$\varphi_v - \varphi_v^2 = C(\mathbf{x}) + d(\mathbf{x}) \int^t \frac{H^2 c_s^2}{(\mu + p) a^3} dt. \qquad (304)$$

$\varphi_v =$ perturbed spatial curvature (φ) in the comoving gauge ($v \equiv 0$) to the second (in fact, to all) order.

Vector-type: in all scales

$$a^4 (\mu + p) v_\alpha^{(v)} = L_\alpha(\mathbf{x}) + \text{decaying mode}. \qquad (305)$$

Tensor-type: in super-horizon scale

$$C_{\alpha\beta}^{(t)} = \text{constant} + \text{decaying mode}. \qquad (306)$$

★ Non-transient solutions in expanding phase are **conserved!**

$$\Phi \equiv \begin{cases} \varphi_v & \text{scalar} \\ a^4 (\mu + p)\, v_\alpha^{(v)} & \text{vector} \\ C_{\alpha\beta}^{(t)} & \text{tensor} \end{cases} \qquad (307)$$

5. Zero-pressure fluid [51, 32]

Newtonian:

Combining (59,60,61):

$$\dot{\delta} + \frac{1}{a}\nabla \cdot \mathbf{u} = -\frac{1}{a}\nabla \cdot (\delta \mathbf{u}), \tag{308}$$

$$\dot{\mathbf{u}} + H\mathbf{u} + \frac{1}{a}\nabla\delta\Phi = -\frac{1}{a}\mathbf{u}\cdot\nabla\mathbf{u}, \tag{309}$$

$$\frac{1}{a^2}\nabla^2\delta\Phi = 4\pi G\varrho\delta, \tag{310}$$

we can derive (65):

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\varrho\delta = -\frac{1}{a^2}\left[a\nabla\cdot(\delta\mathbf{u})\right]^{\cdot} + \frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}). \tag{311}$$

★ These equations are valid to **fully nonlinear order**!

Relativistic:

$$\ddot{\delta}_v + 2H\dot{\delta}_v - 4\pi G\bar{\mu}\delta_v = -\frac{1}{a^2}\left[a\nabla\cdot(\delta_v\mathbf{u})\right]^{\cdot} + \frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) + \dot{C}^{(t)}_{\alpha\beta}\left(\frac{2}{a}\nabla^\alpha u^\beta + \dot{C}^{(t)\alpha\beta}\right). \tag{312}$$

★ This equation is valid to the **second-order**!

To the linear-order we have $\mathbf{u} \equiv -\nabla v_\chi$.

A proof [32]

Fully nonlinear covariant equations:

Take the comoving gauge $\tilde{u}_\alpha = 0$: **only** in this gauge the zero-pressure condition implies vanishing pressures! The covariant equations (18,12) give:

$$\dot{\tilde{\mu}} + \tilde{\mu}\tilde{\theta} = 0, \tag{313}$$

$$\dot{\tilde{\theta}} + \frac{1}{3}\tilde{\theta}^2 + \tilde{\sigma}^{ab}\tilde{\sigma}_{ab} + 4\pi G\tilde{\mu} - \Lambda = 0. \tag{314}$$

By combining

$$\left(\frac{\dot{\tilde{\mu}}}{\tilde{\mu}}\right)^{\dot{}} - \frac{1}{3}\left(\frac{\dot{\tilde{\mu}}}{\tilde{\mu}}\right)^2 - \tilde{\sigma}^{ab}\tilde{\sigma}_{ab} - 4\pi G\tilde{\mu} + \Lambda = 0. \tag{315}$$

To the second-order perturbation:

By **identifying**

$$\delta\mu_v \equiv \delta\varrho, \quad \delta\theta_v \equiv \frac{1}{a}\nabla\cdot\mathbf{u}, \tag{316}$$

(313,314) give

$$\dot{\delta} + \frac{1}{a}\nabla\cdot\mathbf{u} = -\frac{1}{a}\nabla\cdot(\delta\mathbf{u}), \tag{317}$$

$$\frac{1}{a}\nabla\cdot\left(\dot{\mathbf{u}} + \frac{\dot{a}}{a}\mathbf{u}\right) + 4\pi G\mu\delta = -\frac{1}{a^2}\nabla(\mathbf{u}\cdot\nabla\mathbf{u}) - \dot{C}^{(t)\alpha\beta}\left(\frac{2}{a^2}u_{\alpha,\beta} + \dot{C}^{(t)}_{\alpha\beta}\right). \tag{318}$$

Combining (317,318) or (315) gives (312).

5.1 Second-order perturbations: [32]

Relativistic-Newtonian correspondence

Background world model:

Relativistic (Friedmann 1922) vs. Newtonian (Milne 1934)

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\varrho - \frac{\text{const.}}{a^2}, \quad \varrho \propto a^{-3}. \tag{319}$$

Linear perturbation:

Relativistic (Lifshitz 1946) vs. Newtonian (Bonner 1957)

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\varrho\delta = 0. \tag{320}$$

Weakly nonlinear perturbation:

Newtonian (Peebles 1980) vs. Relativistic (Noh-Hwang 2004)

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\varrho\delta = -\frac{1}{a^2}\left[a\nabla\cdot(\delta\mathbf{u})\right]^{\cdot} + \frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) + \dot{C}^{(t)\alpha\beta}\left(\frac{2}{a}u_{\alpha,\beta} + \dot{C}^{(t)}_{\alpha\beta}\right). \tag{321}$$

Except for the gravitational wave contribution, the relativistic zero-pressure fluid perturbed to second order in a flat Friedmann background **coincides exactly** with the Newtonian system.

> *"the linear perturbations are so surprisingly simple that a perturbation analysis accurate to second order may be feasible using the methods of Hawking (1966)."*
>
> Sachs and Wolfe (1967)

5.2 Third-order perturbations: [33]

Pure relativistic corrections

To the third order we identify:

$$\delta\mu_v \equiv \delta\varrho, \quad \delta\theta_v \equiv \frac{1}{a}\nabla\cdot\mathbf{u}. \tag{322}$$

For pure scalar-type perturbation (313,314) give:

$$\dot{\delta}_v + \frac{1}{a}\nabla\cdot\mathbf{u} = -\frac{1}{a}\nabla\cdot(\delta_v\mathbf{u}) + \frac{1}{a}\left[2\varphi_v\mathbf{u} - \nabla\left(\Delta^{-1}X\right)\right]\cdot\nabla\delta_v, \tag{323}$$

$$\frac{1}{a}\nabla\cdot(\dot{\mathbf{u}} + H\mathbf{u}) + 4\pi G\mu\delta_v = -\frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) + \frac{4}{a^2}\nabla\cdot\left[\varphi_v\left(\mathbf{u}\cdot\nabla\mathbf{u} - \frac{1}{3}\mathbf{u}\nabla\cdot\mathbf{u}\right)\right]$$
$$-\frac{2}{3a^2}\varphi_v\mathbf{u}\cdot\nabla(\nabla\cdot\mathbf{u}) - \frac{\Delta}{a^2}\left[\mathbf{u}\cdot\nabla\left(\Delta^{-1}X\right)\right] + \frac{1}{a^2}\mathbf{u}\cdot\nabla X + \frac{2}{3a^2}X\nabla\cdot\mathbf{u}, \tag{324}$$

where

$$X \equiv 2\varphi_v\nabla\cdot\mathbf{u} - \mathbf{u}\cdot\nabla\varphi_v + \frac{3}{2}\Delta^{-1}\nabla\cdot\left[\mathbf{u}\cdot\nabla\left(\nabla\varphi_v\right) + \mathbf{u}\Delta\varphi_v\right]. \tag{325}$$

The first non-vanishing pure relativistic correction terms are φ_v **order higher** than the Newtonian terms.

The CMB temperature anisotropy gives

$$\frac{\delta T}{T} \sim \frac{1}{3}\varphi_\chi \sim \frac{1}{5}\varphi_v \sim \frac{1}{5}C \sim 10^{-5}, \tag{326}$$

in the large-scale limit near horizon scale.

Conclusions in the zero-pressure case: [32, 33]

1. Except for the gravitational wave contribution, equations for the relativistic zero-pressure fluid in a flat Friedmann background *coincide* exactly with the previously known Newtonian equations even to the second-order perturbation.

2. To the second order, we identify the relativistic density and velocity perturbation variables. However, we do *not* have a relativistic variable which corresponds to the Newtonian potential to the second order.

3. We assume a flat Friedmann background but include the cosmological constant, thus our results are relevant to currently favored cosmology.

4. We *expand* the range of applicability of the Newtonian medium without pressure to all cosmological scales including the super-horizon scale.

5. The Newtonian equations are exact to the second order nonlinearity. Thus, non-vanishing terms in the third and higher order are pure relativistic effects.

6. The third-order correction terms in relativistic analysis, thus the pure general relativistic effects, are of φ_v-order higher than the second-order Newtonian terms.

7. The corrections terms are *independent* of the horizon scale and depend only on the linear order gravitational potential (curvature) perturbation strength φ_v.

8. From the temperature anisotropy of CMB we have $\frac{\delta T}{T} \sim \frac{1}{3}\delta\Phi \sim \frac{1}{5}\varphi_v \sim 10^{-5}$.

9. Therefore, one can use the large-scale Newtonian numerical simulation more reliably even as the simulation scale approaches near (and goes beyond) the horizon.

Complimentary methods: [51]

1. The large-scale (long wavelength) approximation or the spatial gradient expansion.
2. Cosmological post-Newtonian formulation.
3. Relativistic Zel'dovich approximation.
4. General (spatially inhomogeneous and anisotropic) solutions near singularity where the large-scale condition is well met.
5. Fitting and averaging.

Future applications: [51]

1. Limit of the linear theory.

   The limit of the linear theory *cannot* be estimated within the linear theory.

2. Quasilinear processes leading to the mode-mode coupling among different scales, as well as among different types of perturbations.

3. The non-Gaussian effects due to nonlinear processes in the quantum generation and in the classical evolution.

4. Fate of fluctuations in the collapsing phase, and possibly through a bounce.

   In the collapsing phase fluctuations grow. The d-mode becomes the growing one.

   As the background reaches singularity the fluctuations inevitably diverge.

5. Nonlinear evolutions in the super-horizon scale.

6. Nonlinear backreaction to the background world model.

"Second order perturbation is closer to the linear theory than to the nonlinear one."

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