Several Mathematical Formulations of Einstein's Gravity and their Applications to Cosmology

J. Hwang (KNU)

"Mini International Workshop on Lagrangian Submanifolds and Related Fields"

Dept. of Math., 2007.12.06

Newton’s Theory

Newton's force law (1687)

Inertial mass

Force

$$\mathbf{F} = m\mathbf{a}.$$

Acceleration

Newton's gravity

Gravitational constant

Gravitational masses

Gravitational force

$$\mathbf{F}_g = -\frac{GMm}{r^2}\hat{\mathbf{r}}.$$

Distance

Poisson's formulation (1812)

Gravitational potential

$$\mathbf{F}_g = -\nabla\Phi,$$

Mass density

$$\nabla^2\Phi = 4\pi G\varrho.$$

Two theories of Gravity

- Newton (1647-1727): "Philosophiae naturalis principia mathematica" (1687)

  "But hitherto I have not been able to discover the cause of those properties of gravity from phaenomena, and I frame no hypotheses; for whatever is not deduced from the phaenomena, is to be called an hypotheses; an hypotheses, whether metaphysical or physical, whether of occult qualities or mechanical, have no place in experimental philosophy. … And to us it is enough that gravity does really exist, and act according to the laws which we have explained, and abundantly serves to account for all the motions of the celestial bodies, and of our sea [sun?]."

- On this regard, Einstein's gravity is no better.

- Einstein (1879-1955): "Die Feldgleichungen der Gravitation" (1915)

  "Let us put $$R_{im} = -\kappa\left(T_{im} - \frac{1}{2} g_{im} T\right)."$$

- In practice, however, Einstein's gravity provides much better perspective.

Newton, I., 1713, *The mathematical principles of natural philosophy*, 2nd edition, Book III, General Scholium; Translated into English by Motte, A. in 1729, 1962 (University of California Press).

Einstein, A., *Preuss. Akad. Wiss. Berlin, Sitzber.*, 844-847 (1915); Translated in Misner, C. W., Thorne, K. S., and Wheeler, J. A., 1973, *Gravitation*, (Freeman and Company) p. 433.

- **<u>Newton's gravity</u>** <u>(1687):</u>

  Non-relativistic (no c)

  - $c \to \infty$ limit of Einstein gravity
  - Action at a distance, violates causality
  - No horizon
  - Static nature

  No strong pressure allowed

  No strong gravity allowed

  No gravitational waves

  Incomplete and inconsistent

- **<u>Einstein's gravity</u>** <u>(1915):</u>

  Relativistic gravity

  Strong gravity, dynamic

  Simplest

- In the zero-pressure limit, the two theories give the same descriptions for the cosmological world model and its linear structures.

Einstein's Gravity

Einstein's equation

Cosmological constant

Ricci tensor

Ricci scalar curvature

Energy-momentum tensor

$$R_{ab} - \frac{1}{2} R g_{ab} = \frac{8\pi G}{c^4} T_{ab} - \Lambda g_{ab}.$$

Metric tensor

Gravitational constant

Speed of light

$$T^{ab}{}_{;b} = 0.$$

Energy-momentum conservation

Covariant derivative

1+3 Approach

Covariant Formulation

Ehlers, J., 1961 *Proceedings of the mathematical-natural science of the Mainz academy of science and literature*, Nr. **11**, 792 (1961), translated in Gen. Rel. Grav. **25**, 1225;

Ellis, G. F. R., 1971 in *General relativity and cosmology*, Proceedings of the international summer school of physics Enrico Fermi course 47, edited by R. K. Sachs (Academic Press, New York), p104;

Ellis, G. F. R., 1973 in *Cargese Lectures in Physics*, edited by E. Schatzmann (Gorden and Breach, New York), p1.

Kinematic quantities

Four vector

Projection tensor

$$u_a u^a = -1 \qquad h_{ab} \equiv g_{ab} + u_a u_b .$$

$$h_a^c h_b^d u_{c;d} \equiv \omega_{ab} + \theta_{ab} = u_{a;b} + \dot{u}_a u_b$$

Shear tensor

Expansion scalar

$$\sigma_{ab} \equiv \theta_{ab} - \tfrac{1}{3}\theta h_{ab} , \qquad \theta \equiv u^a{}_{;a} .$$

Acceleration vector

$$u_{a;b} = \omega_{ab} + \sigma_{ab} + \tfrac{1}{3}\theta h_{ab} - a_a u_b ,$$

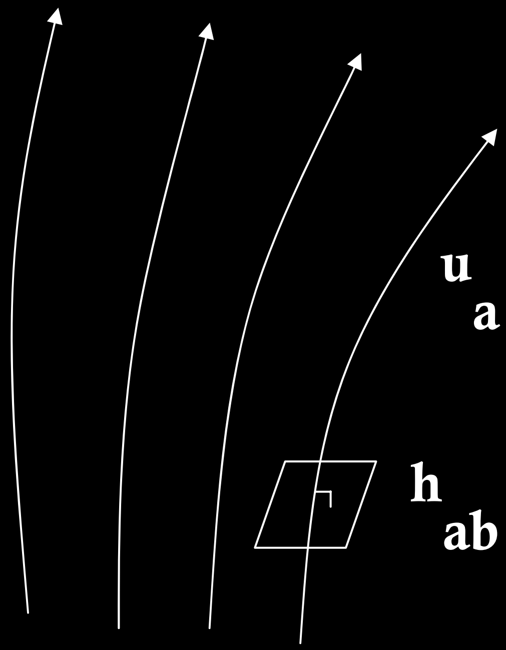
Vorticity tensor

Vorticity vector

$$a_a \equiv \dot{u}_a \equiv u_{a;b} u^b , \qquad \omega^a \equiv \tfrac{1}{2}\eta^{abcd} u_b \omega_{cd} , \qquad \omega_{ab} = \eta_{abcd} \omega^c u^d ,$$

http://bh.knu.ac.kr/~jchan/paper/1990-ApJ-Covariant.pdf

1+3



- Ehlers etal (1961)
- Spacetime covariant
- Useful in cosmology

Energy-momentum tensor

Four vector

Projection tensor

$$T_{ab} = \mu u_a u_b + p h_{ab} + q_a u_b + q_b u_a + \pi_{ab}$$

Energy density

Pressure

Flux vector

Anisotropic stress tensor

$$\mu = T_{ab} u^a u^b \qquad p = \tfrac{1}{3} T_{ab} h^{ab} \qquad q_a = -T_{cd} u^c h^d_a \qquad \pi_{ab} = T_{cd} h^c_a h^d_b - p h_{ab}.$$

http://bh.knu.ac.kr/~jchan/paper/1990-CQG-GGT1.pdf

Covariant equations

$$R_{ab} = T_{ab} - \tfrac{1}{2} T g_{ab} + \Lambda g_{ab} \, .$$

The propagation and constraint equations follow from the Ricci identity

$$u_{a;b;c} - u_{a;c;b} = u_d R^d{}_{abc} \, .$$

The three (trace, antisymmetric, and trace-less symmetric parts) propagation equations are the following.

Raychaudhuri equation:

$$\boxed{\dot{\theta} + \tfrac{1}{3}\theta^2 - a^a{}_{;a} + 2(\sigma^2 - \omega^2) + \tfrac{1}{2}(\mu + 3p) - \Lambda = 0 \, .}$$

Vorticity propagation:

$$h^a_b \dot{\omega}^b + \tfrac{2}{3}\theta\omega^a = \sigma^a_b \omega^b + \tfrac{1}{2}\eta^{abcd} u_b a_{c;d} \, .$$

Shear propagation:

$$h^f_a h^g_b \dot{\sigma}_{fg} - h^f_a h^g_b a_{(f;g)} - a_a a_b + \omega_a \omega_b + \sigma_{af}\sigma^f_b + \tfrac{2}{3}\theta\sigma_{ab} + h_{ab}(-\tfrac{1}{3}\omega^2 - \tfrac{2}{3}\sigma^2 + \tfrac{1}{3}a^c{}_{;c}) + E_{ab} - \tfrac{1}{2}\pi_{ab} = 0 \, .$$

The three constraint equations are the following:

(*ADM*) *momentum constraint* (or [0, α] field equation):

$$h^e_b(\omega^{bc}{}_{;c} - \sigma^{bc}{}_{;c} + \tfrac{2}{3}\theta^{;b}) + (\omega^e{}_b + \sigma^e{}_b)a^b = q^e \, .$$

Vorticity constraint:

$$\omega^a{}_{;a} = 2\omega^b a_b \, .$$

H constraint:

$$H_{ad} = 2a_{(a}\omega_{d)} - h^t_a h^s_d(\omega_{(t}{}^{b;c} + \sigma_{(t}{}^{b;c})\eta_{s)fbc} u^f \, .$$

Energy conservation:

$$-u_a T^{ab}{}_{;b} = \boxed{\dot{\mu} + (\mu + p + \pi)\theta + \pi^{ab}\sigma_{ab} + q^a{}_{;a} + q^a a_a = 0 \, .}$$

Momentum conservation:

$$h_{ab} T^{bc}{}_{;c} = (\mu + p + \pi)a_a + h^c_a[(p + \pi)_{,c} + \pi^b_{c;b} + \dot{q}_c] + (\omega_a{}^b + \sigma_a{}^b + 4/3\theta h^b_a)q_b = 0 \, .$$

Weyl tensor

Antisymmetrization symbol

Electric part　　Weyl (conformal) tensor　　Magnetic part

$$C^{ab}{}_{cd} \equiv R^{ab}{}_{cd} - 2g^{[a}_{[c}R^{b]}_{d]} + \tfrac{1}{3}Rg^{[a}_{[c}g^{b]}_{d]} \,,$$

$$E_{ac} \equiv C_{abcd}u^b u^d \,, \qquad H_{ac} \equiv \tfrac{1}{2}\eta_{ab}{}^{gh}C_{ghcd}u^b u^d \,,$$

$$C^{abcd} = (\eta^{ab}{}_{pq}\eta^{cd}{}_{rs} + g^{ab}{}_{pq}g^{cd}{}_{rs})u^p u^r E^{qs} - (\eta^{ab}{}_{pq}g^{cd}{}_{rs} + g^{ab}{}_{pq}\eta^{cd}{}_{rs})u^p u^r H^{qs} \,,$$

$$g_{abcd} \equiv g_{ac}g_{bd} - g_{ad}g_{bc} \,, \qquad \eta^{abcd} = \eta^{[abcd]} \,, \qquad \eta^{1234} = (-g)^{-1/2} \,,$$

The Bianchi identities $R_{ab[cd;e]} = 0$, can be expressed as

$$C_{abc}{}^{d}{}_{;d} = R_{c[a;b]} - \tfrac{1}{6}g_{c[a}R_{;b]} \,,$$

and from this we can derive the following four quasi-Maxwellian equations.

div E:

$$h^t_a E^{as}{}_{;d}h^d_s - \eta^{tbpq}u_b\sigma^d_p H_{qd} + 3H^t_s\omega^s = \tfrac{1}{3}h^{tb}\mu_{,b} - \tfrac{1}{2}h^t_c\pi^{cb}{}_{;b} - \tfrac{3}{2}\omega^t{}_b q^b + \tfrac{1}{2}\sigma^t_b q^b + \tfrac{1}{2}\pi^t_a a^a - \tfrac{1}{3}\theta q^t \,.$$

div H:

$$h^t_a H^{as}{}_{;d}h^d_s + \eta^{tbpq}u_b\sigma^d_p E_{qd} - 3E^t_s\omega^s = (\mu + p)\omega^t + \tfrac{1}{2}\eta^{tbef}u_b q_{[e;f]} + \tfrac{1}{2}\eta^{tbef}u_b\pi_{ec}(\omega^c{}_f + \sigma^c{}_f) \,.$$

$\dot{E}$:

$$h^m_a h^t_c \dot{E}^{ac} + h^{(m}_a\eta^{t)rsd}u_r H^a_{s;d} - 2H^{(t}_q\eta^{m)bpq}u_b a_p + h^{tm}\sigma^{ab}E_{ab} + \theta E^{mt} - 3E^{(m}_s\sigma^{t)s} - E^{(m}_s\omega^{t)s}$$
$$= -\tfrac{1}{2}(\mu + p)\sigma^{tm} - a^{(t}q^{m)} - \tfrac{1}{2}h^t_a h^m_c q^{(a;c)} - \tfrac{1}{2}h^t_a h^m_c\dot{\pi}^{ac} - \tfrac{1}{2}\pi^{b(m}\omega_b{}^{t)} - \tfrac{1}{2}\pi^{b(m}\sigma^{t)}_b - \tfrac{1}{6}\pi^{tm}\theta + \tfrac{1}{6}(q^a{}_{;a} + a_a q^a + \pi^{ab}\sigma_{ab})h^{mt} \,.$$

$\dot{H}$:

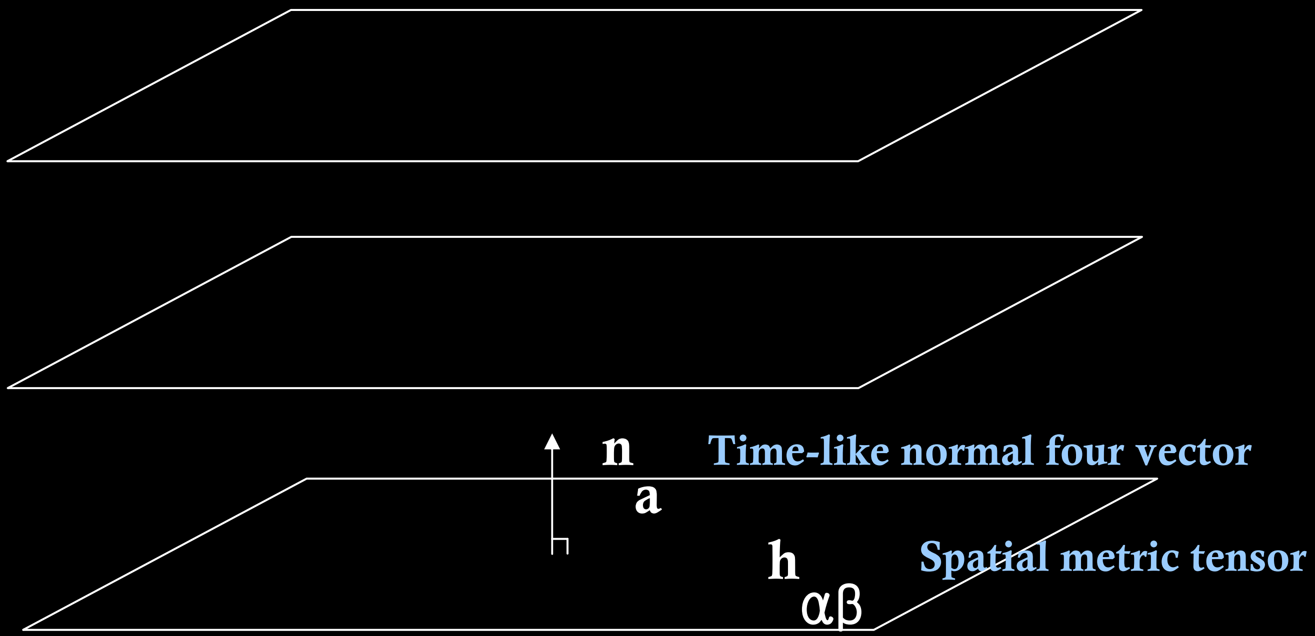
$$h^m_a h^t_c \dot{H}^{ac} - h^{(m}_a\eta^{t)rsd}u_r E^a_{s;d} + 2E^{(t}_q\eta^{m)bpq}u_b a_p + h^{tm}\sigma^{ab}H_{ab} + \theta H^{mt} - 3H^{(m}_s\sigma^{t)s} - H^{(m}_s\omega^{t)s}$$
$$= \tfrac{1}{2}\sigma^{(t}_e\eta^{m)bef}u_b q_f - \tfrac{1}{2}h^{(t}_c\eta^{m)bef}u_b\pi^c_{e;f} + \tfrac{1}{2}[h^{mt}\omega_c q^c - 3\omega^{(m}q^{t)}] \,.$$

3+1 Approach

ADM Formulation

Arnowitt R, Deser S and Misner C W, 1962 in *Gravitation: an introduction to current research*, edited by L. Witten (Wiley, New York) p. 227.

3+1



- Arnowitt-Deser-Misner (1962)
- Canonical quantization
- Useful in numerical relativity

Lapse function

Shift vector

Three-space metric

$$\tilde{g}_{00} \equiv -N^2 + N^\alpha N_\alpha, \quad \tilde{g}_{0\alpha} \equiv N_\alpha, \quad \tilde{g}_{\alpha\beta} \equiv h_{\alpha\beta},$$

$$\tilde{g}^{00} = -N^{-2}, \quad \tilde{g}^{0\alpha} = N^{-2} N^\alpha,$$

$$\tilde{g}^{\alpha\beta} = h^{\alpha\beta} - N^{-2} N^\alpha N^\beta,$$

Extrinsic curvature

Covariant derivative based on $h_{\alpha\beta}$

$$K_{\alpha\beta} \equiv \frac{1}{2N}(N_{\alpha:\beta} + N_{\beta:\alpha} - h_{\alpha\beta,0}), \quad K \equiv h^{\alpha\beta} K_{\alpha\beta},$$

$$\bar{K}_{\alpha\beta} \equiv K_{\alpha\beta} - \frac{1}{3} h_{\alpha\beta} K,$$

http://bh.knu.ac.kr/~jchan/paper/2004-PRD-Second-order.pdf

Intrinsic curvature

$$R^{(h)\alpha}{}_{\beta\gamma\delta} \equiv \Gamma^{(h)\alpha}{}_{\beta\delta,\gamma} - \Gamma^{(h)\alpha}{}_{\beta\gamma,\delta} + \Gamma^{(h)\epsilon}{}_{\beta\delta}\Gamma^{(h)\alpha}{}_{\gamma\epsilon} - \Gamma^{(h)\epsilon}{}_{\beta\gamma}\Gamma^{(h)\alpha}{}_{\delta\epsilon},$$

$$R^{(h)}_{\alpha\beta} \equiv R^{(h)\gamma}{}_{\alpha\gamma\beta}, \qquad R^{(h)} \equiv h^{\alpha\beta} R^{(h)}_{\alpha\beta},$$

$$\overline{R^{(h)}_{\alpha\beta}} \equiv R^{(h)}_{\alpha\beta} - \frac{1}{3} h_{\alpha\beta} R^{(h)}.$$

Normal four-vector

$$\tilde{n}_0 \equiv -N, \quad \tilde{n}_\alpha \equiv 0, \quad \tilde{n}^0 = N^{-1}, \quad \tilde{n}^\alpha = -N^{-1} N^\alpha .$$

Energy **Momentum**

$$E \equiv \tilde{n}_a \tilde{n}_b \tilde{T}^{ab}, \quad J_\alpha \equiv -\tilde{n}_b \tilde{T}^b_\alpha, \quad S_{\alpha\beta} \equiv \tilde{T}_{\alpha\beta},$$

$$S \equiv h^{\alpha\beta} S_{\alpha\beta}, \quad \overline{S}_{\alpha\beta} \equiv S_{\alpha\beta} - \frac{1}{3} h_{\alpha\beta} S,$$

Pressure **Stress**

ADM equations

(0,0) component:

Energy constraint equation

$$R^{(h)} = \overline{K}^{\alpha\beta}\overline{K}_{\alpha\beta} - \frac{2}{3}K^2 + 16\pi G E + 2\Lambda, \tag{8}$$

where Λ is the cosmological constant.

(0,α) component:

Momentum constraint equation

$$\overline{K}^{\beta}_{\alpha:\beta} - \frac{2}{3}K_{,\alpha} = 8\pi G J_{\alpha}. \tag{9}$$

(α,β) component:

Trace of ADM propagation equation

$$K_{,0}N^{-1} - K_{,\alpha}N^{\alpha}N^{-1} + N^{:\alpha}{}_{\alpha}N^{-1} - \overline{K}^{\alpha\beta}\overline{K}_{\alpha\beta} - \frac{1}{3}K^2 - 4\pi G(E+S) + \Lambda = 0. \tag{10}$$

Trace-free ADM propagation equation

$$\overline{K}^{\alpha}_{\beta,0}N^{-1} - \overline{K}^{\alpha}_{\beta:\gamma}N^{\gamma}N^{-1} + \overline{K}_{\beta\gamma}N^{\alpha:\gamma}N^{-1} - \overline{K}^{\alpha}_{\gamma}N^{\gamma}{}_{:\beta}N^{-1} = K\overline{K}^{\alpha}_{\beta} - \left(N^{:\alpha}{}_{\beta} - \frac{1}{3}\delta^{\alpha}_{\beta}N^{:\gamma}{}_{\gamma}\right)N^{-1} + \overline{R^{(h)\alpha}}_{\beta} - 8\pi G\overline{S}^{\alpha}_{\beta}.$$

Energy conservation equation

$$E_{,0}N^{-1}-E_{,\alpha}N^{\alpha}N^{-1}-K\left(E+\frac{1}{3}S\right)-\overline{S^{\alpha\beta}}\overline{K_{\alpha\beta}}$$
$$+N^{-2}(N^2J^{\alpha})_{:\alpha}=0. \tag{12}$$

Momentum conservation equation

$$J_{\alpha,0}N^{-1}-J_{\alpha:\beta}N^{\beta}N^{-1}-J_{\beta}N^{\beta}{}_{:\alpha}N^{-1}-KJ_{\alpha}+EN_{,\alpha}N^{-1}$$
$$+S^{\beta}_{\alpha:\beta}+S^{\beta}_{\alpha}N_{,\beta}N^{-1}=0. \tag{13}$$

Action Formulation

Einstein-Hilbert action

$$S = \int \sqrt{-g} \left[\frac{c^4}{16\pi G} (R - 2\Lambda) + L_m \right] d^4x,$$

$$\delta\left(\sqrt{-g} L_m\right) \equiv \frac{1}{2} T^{ab} \delta g_{ab},$$

$$G_{ab} = \frac{8\pi G}{c^4} T_{ab} - \Lambda g_{ab},$$

Einstein tensor

Hilbert, D. (1915) *Die Grundlagen der Physik*, Konigl. Gesell. d. Wiss. Gottingen, Nachr. Math.-Phys. Kl. 395-407
http://einstein-annalen.mpiwg-berlin.mpg.de/related_texts/relativity_rev/hilbert

Cosmology

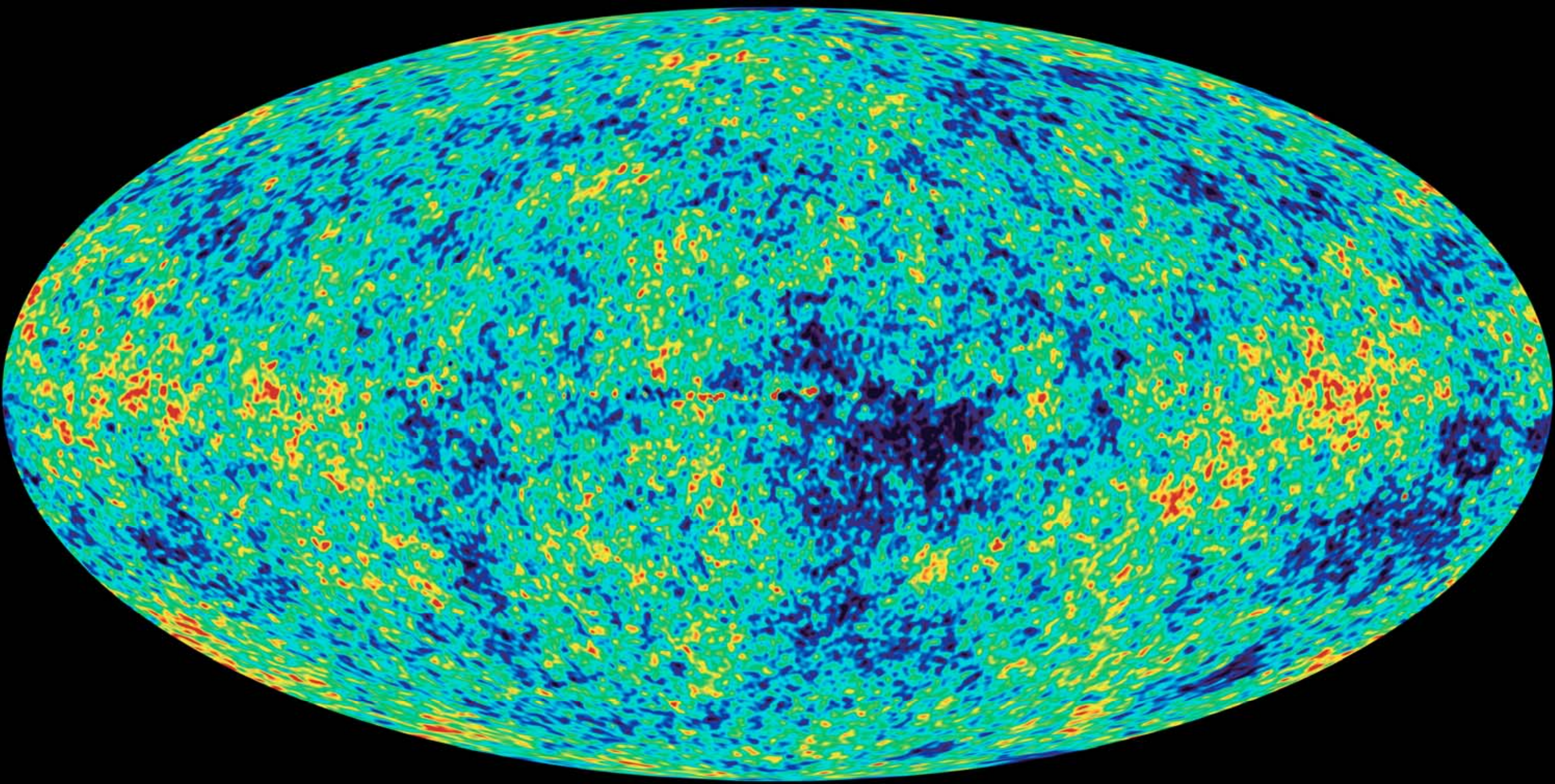
History

Cosmology:

- 1917 Einstein: static world model
- 1922 Friedmann: dynamic world model
- 1929 Hubble: expansion
- 1965 Penzias-Wilson: CMB
- 1981 Inflation (early acceleration) hypothesis
- 1992 COBE: CMB temperature anisotropies
- 1997 Recent acceleration

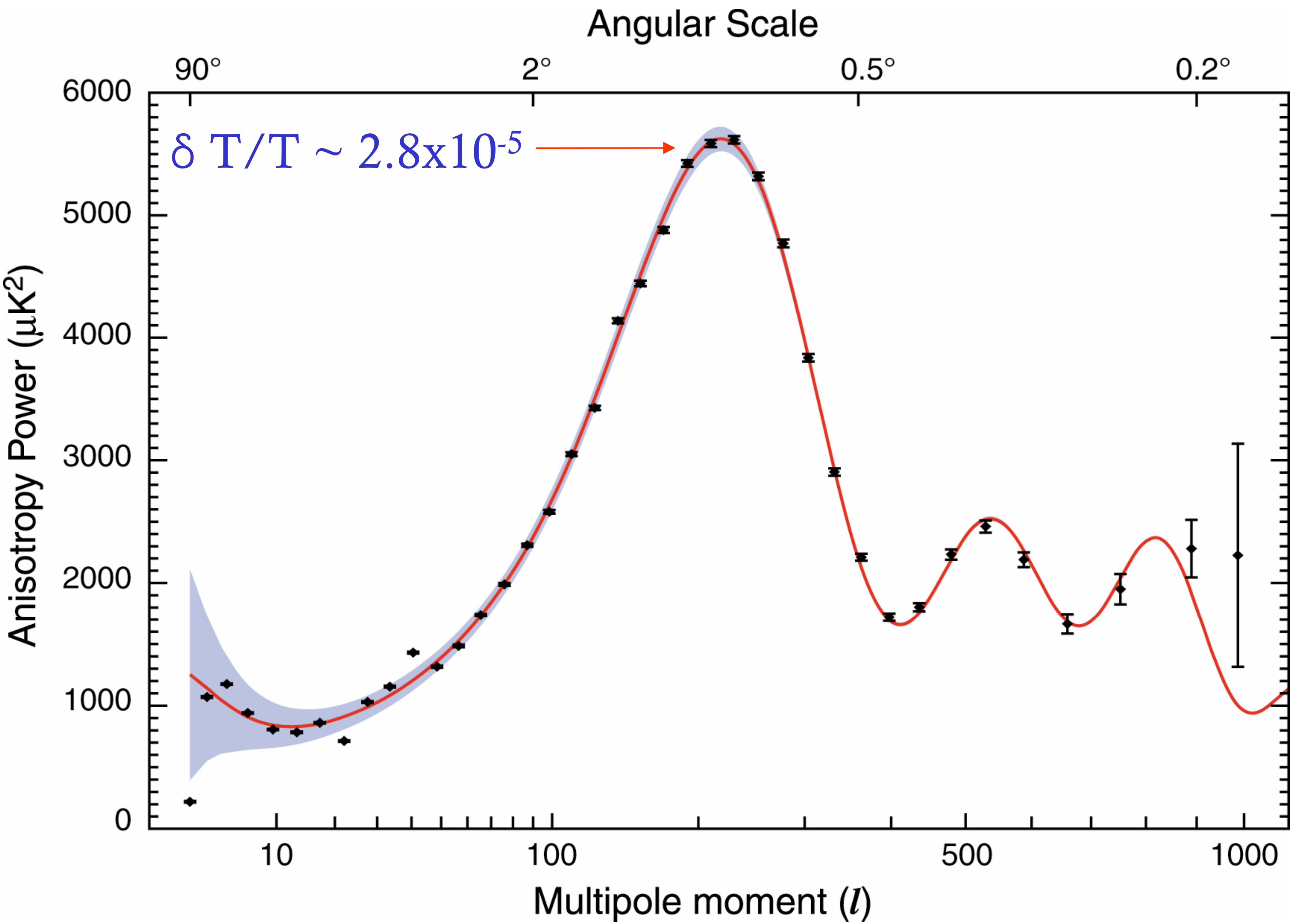
Observations

CMB: linear structure $\delta T/T \sim 10^{-5}$



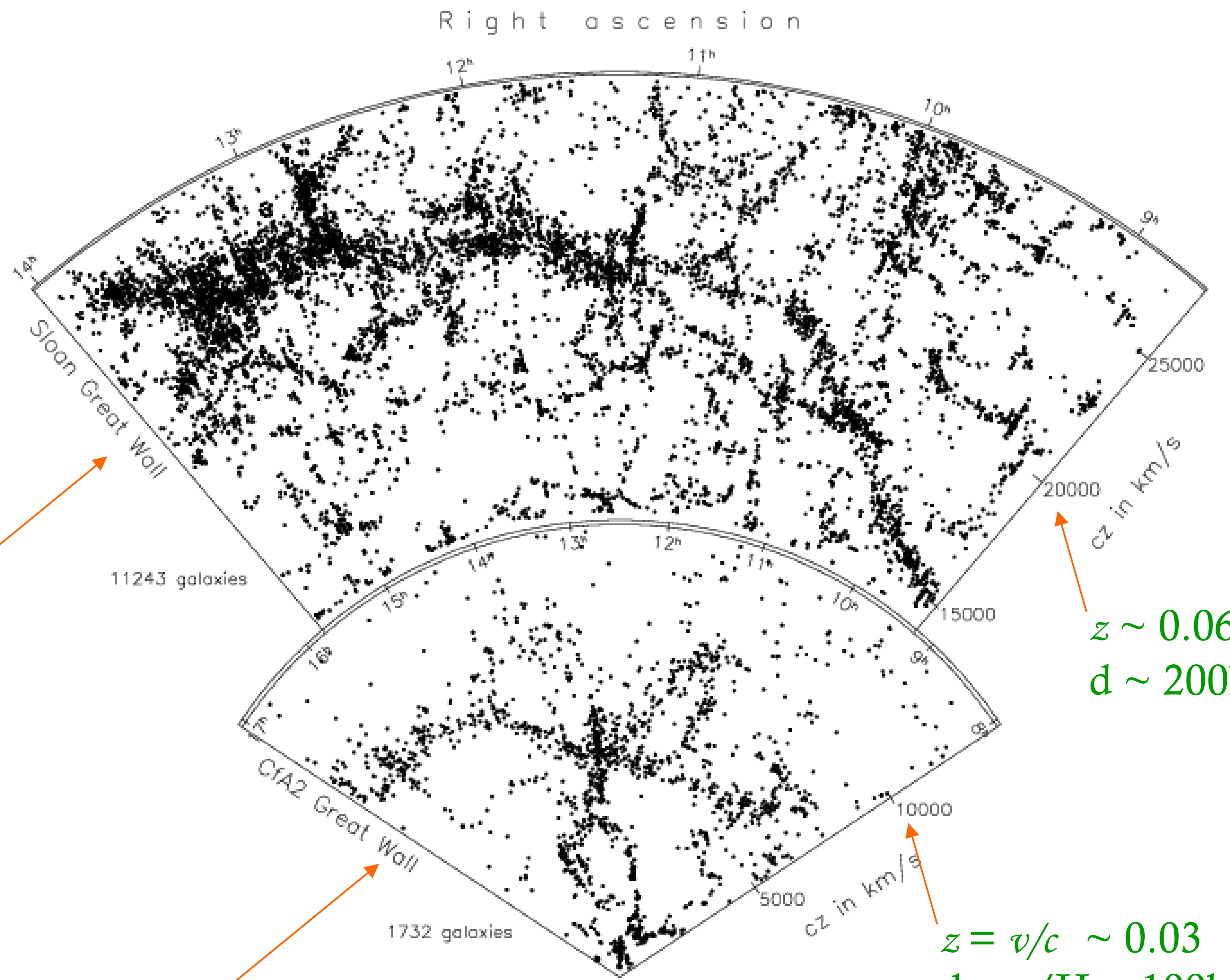
http://map.gsfc.nasa.gov/m_or.html

WMAP Temperature anisotropy power spectrum



http://map.gsfc.nasa.gov/m_mm.html

LSS: non-linear structure



SDSS

de Lapparent *etal.* (1986)

$z \sim 0.06$
d ~ 200h^{-1}Mpc

$z = v/c \sim 0.03$
d = v/H ~ 100h^{-1}Mpc

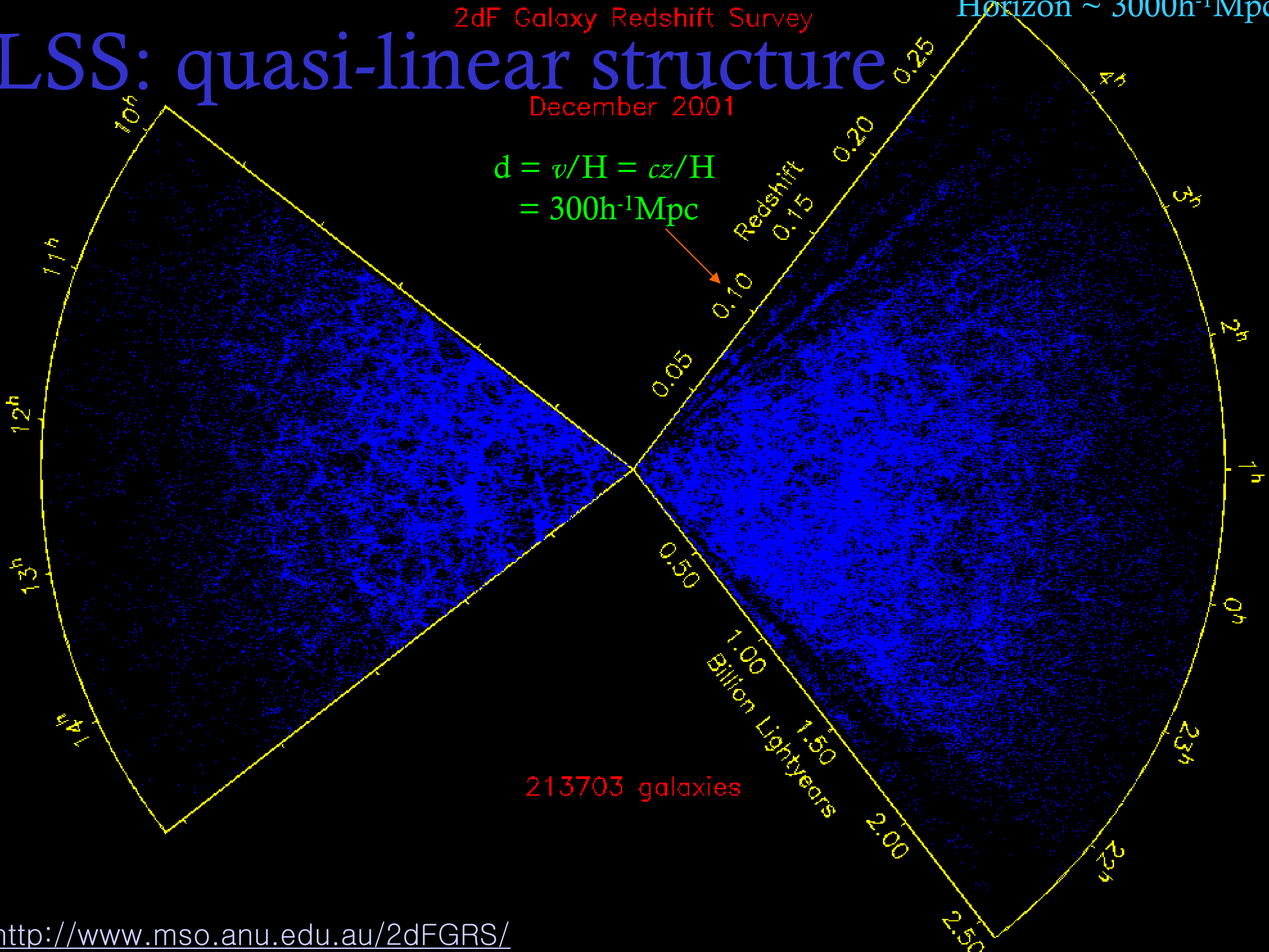
http://www.sdss.org/

LSS: quasi-linear structure

Horizon ~ $3000h^{-1}$Mpc

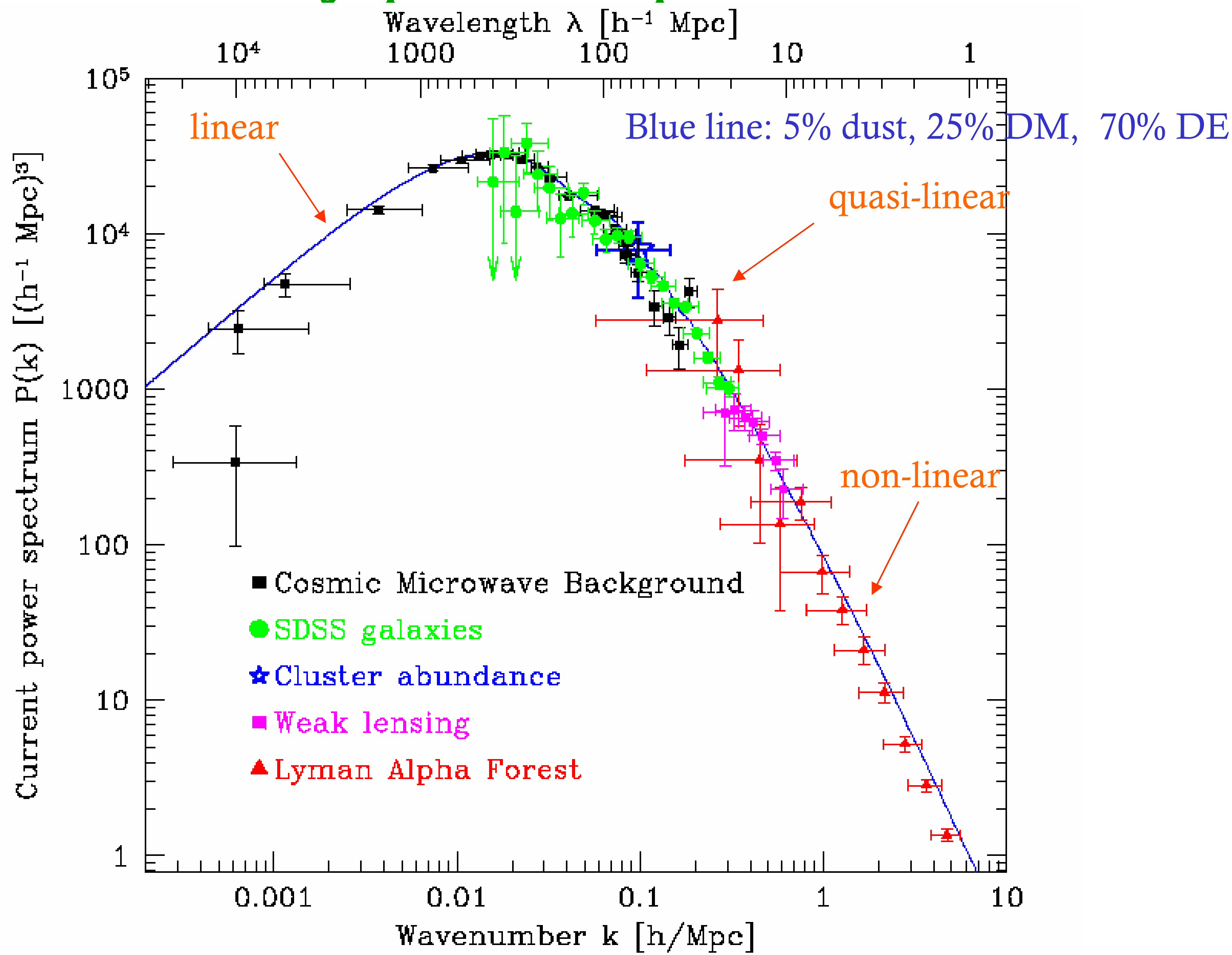
$d = v/H = cz/H$

$= 300h^{-1}$Mpc



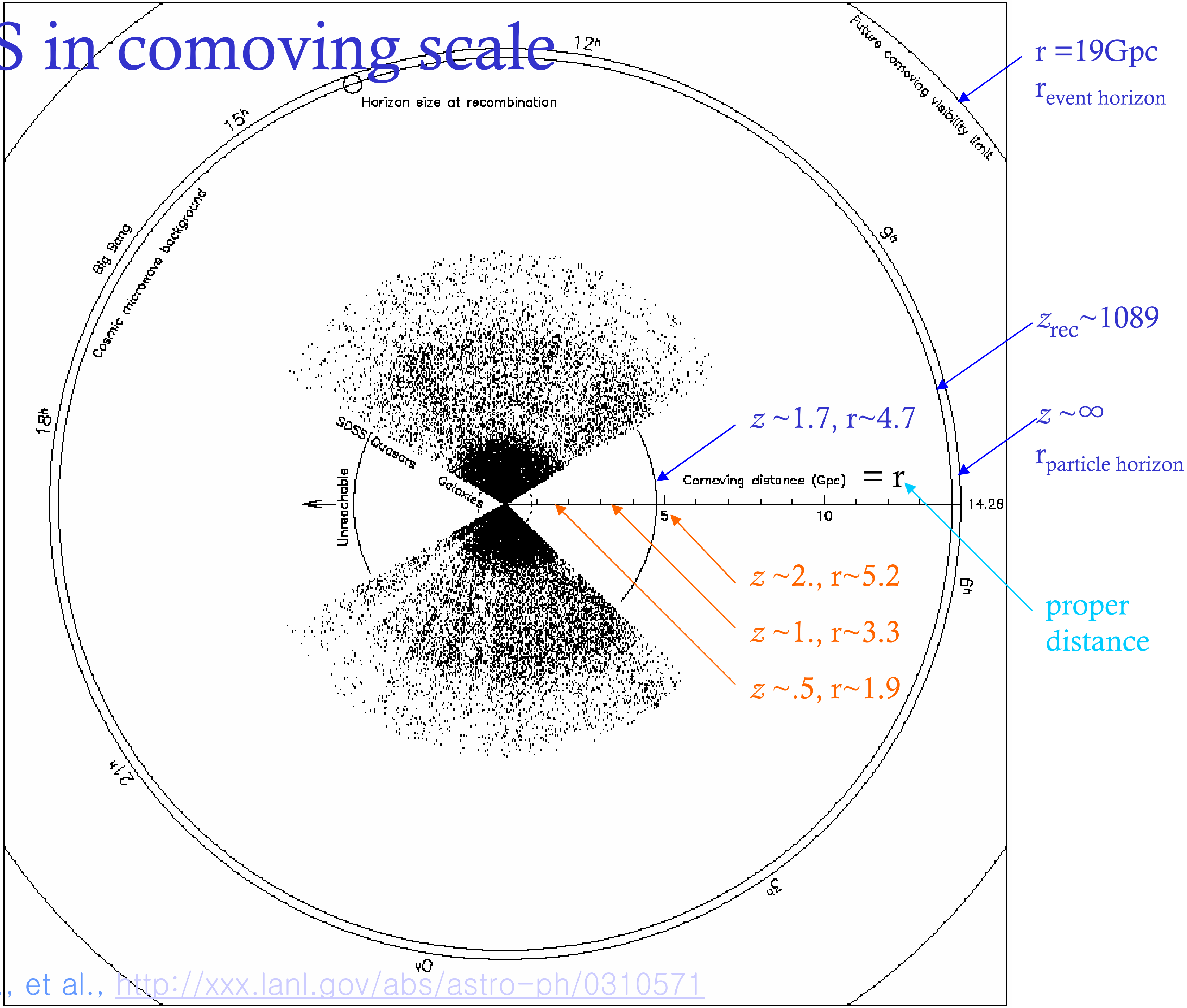
http://www.mso.anu.edu.au/2dFGRS/

Density power spectrum



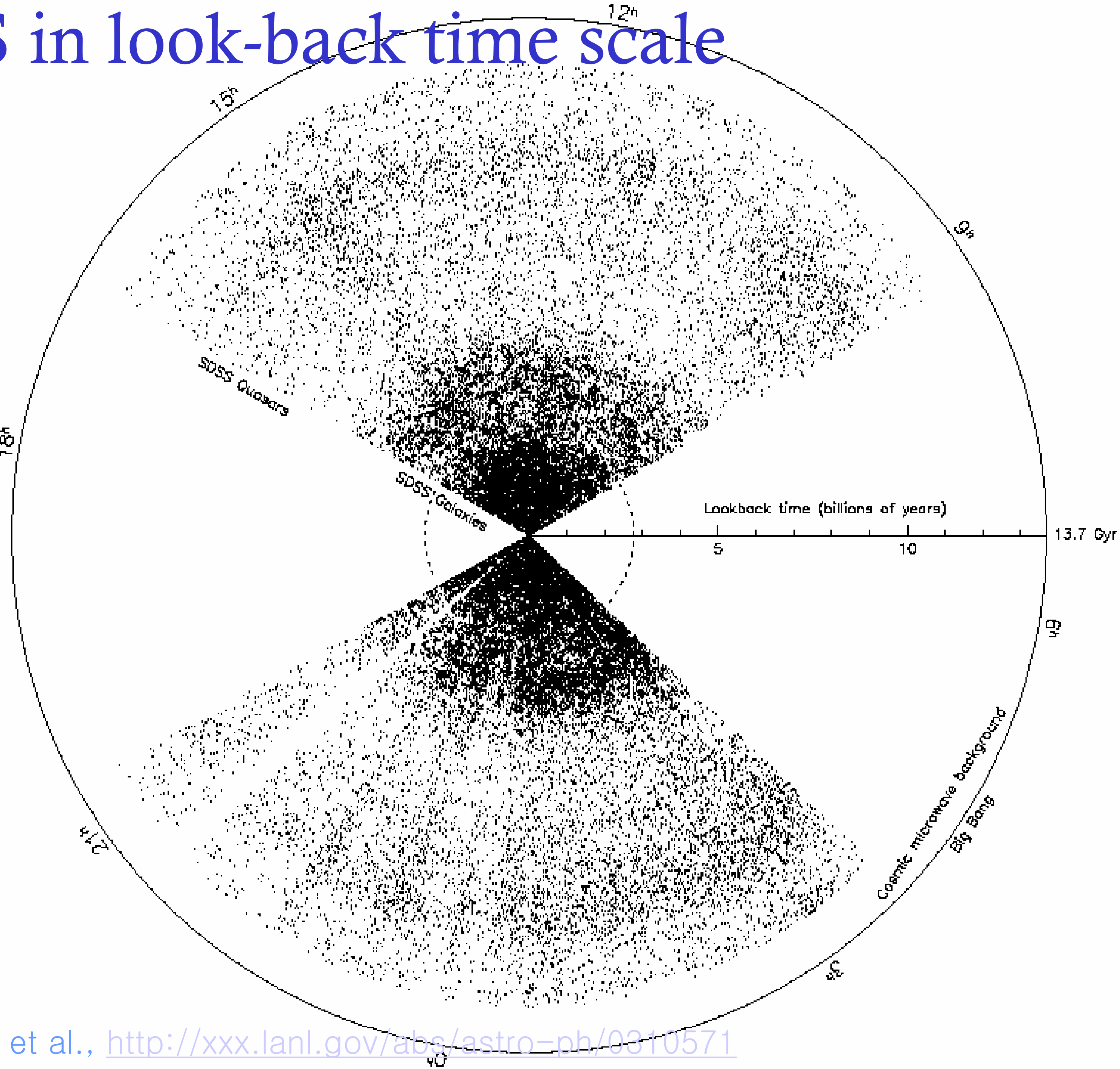
Tegmark, M., et al, http://xxx.lanl.gov/pdf/astro-ph/0207047

SDSS in comoving scale



Gott, J. R., et al., http://xxx.lanl.gov/abs/astro-ph/0310571

SDSS in look-back time scale



Gott, J. R., et al., http://xxx.lanl.gov/abs/astro-ph/0310571

Theoretical World Models

Four ingredients (assumptions):

1. Gravity: Einstein gravity or generalized gravity
2. Spatial geometry: homogeneous and isotropic, or more complicated geometries.
3. Matter contents: dust, radiation, fields, and others.
4. Topology (global geometry): undetermined in the gravity level.

Robertson-Walker metric

Scale factor

$$ds^2 = -c^2 dt^2 + a^2(t) g^{(3)}_{\alpha\beta} dx^\alpha dx^\beta .$$

$$\begin{aligned} g^{(3)}_{\alpha\beta} dx^\alpha dx^\beta &= \frac{dr^2}{1 - Kr^2} + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \\ &= \frac{1}{(1 + \frac{K}{4}\bar{r}^2)^2} (dx^2 + dy^2 + dz^2) \\ &= d\bar{\chi}^2 + \left[\frac{1}{\sqrt{K}} \sin(\sqrt{K}\bar{\chi}) \right]^2 (d\theta^2 + \sin^2\theta d\phi^2), \end{aligned} \tag{2}$$

Sign of the spatial curvature

where we have

$$r \equiv \frac{\bar{r}}{1 + \frac{K}{4}\bar{r}^2}, \qquad \bar{r} \equiv \sqrt{x^2 + y^2 + z^2}, \qquad \bar{\chi} \equiv \int^r \frac{dr}{\sqrt{1 - Kr^2}}. \tag{3}$$

http://bh.knu.ac.kr/~jchan/paper/2005-PRD-GGT.pdf

Friedmann equations

Nonlinear order

ADM E constraint

$$H^2 = \frac{8\pi G}{3}\mu - \frac{K}{a^2} + \frac{\Lambda}{3} + \frac{1}{3}(\cancel{\sigma^2} - \cancel{\omega^2}) , \qquad (22)$$

E conservation

$$\dot{\mu} = -3H(\mu + p) - \cancel{\pi^{ab}}\sigma_{ab} , \qquad (23)$$

Raychaudhury eq.

$$\dot{H} + H^2 = -\frac{4\pi G}{3}(\mu + 3p) + \frac{\Lambda}{3} + \frac{1}{3}\cancel{\nabla_a} a^a - \frac{2}{3}(\cancel{\sigma^2} - \cancel{\omega^2}) , \qquad (24)$$

Spatial gradient

Mom conservation

$$a_a = -\frac{1}{\mu + p} h_a^b(\nabla_b p + \nabla_c \pi_b^c) \qquad (25)$$

$$H^2 = \frac{8\pi G}{3}\mu - \frac{K}{a^2} + \frac{\Lambda}{3} ,$$

$$\dot{\mu} = -3H(\mu + p) ,$$

$$\dot{H} + H^2 = -\frac{4\pi G}{3}(\mu + 3p) + \frac{\Lambda}{3} ,$$

$$\theta(\mathbf{x}, t) \equiv 3H(\mathbf{x}, t),$$

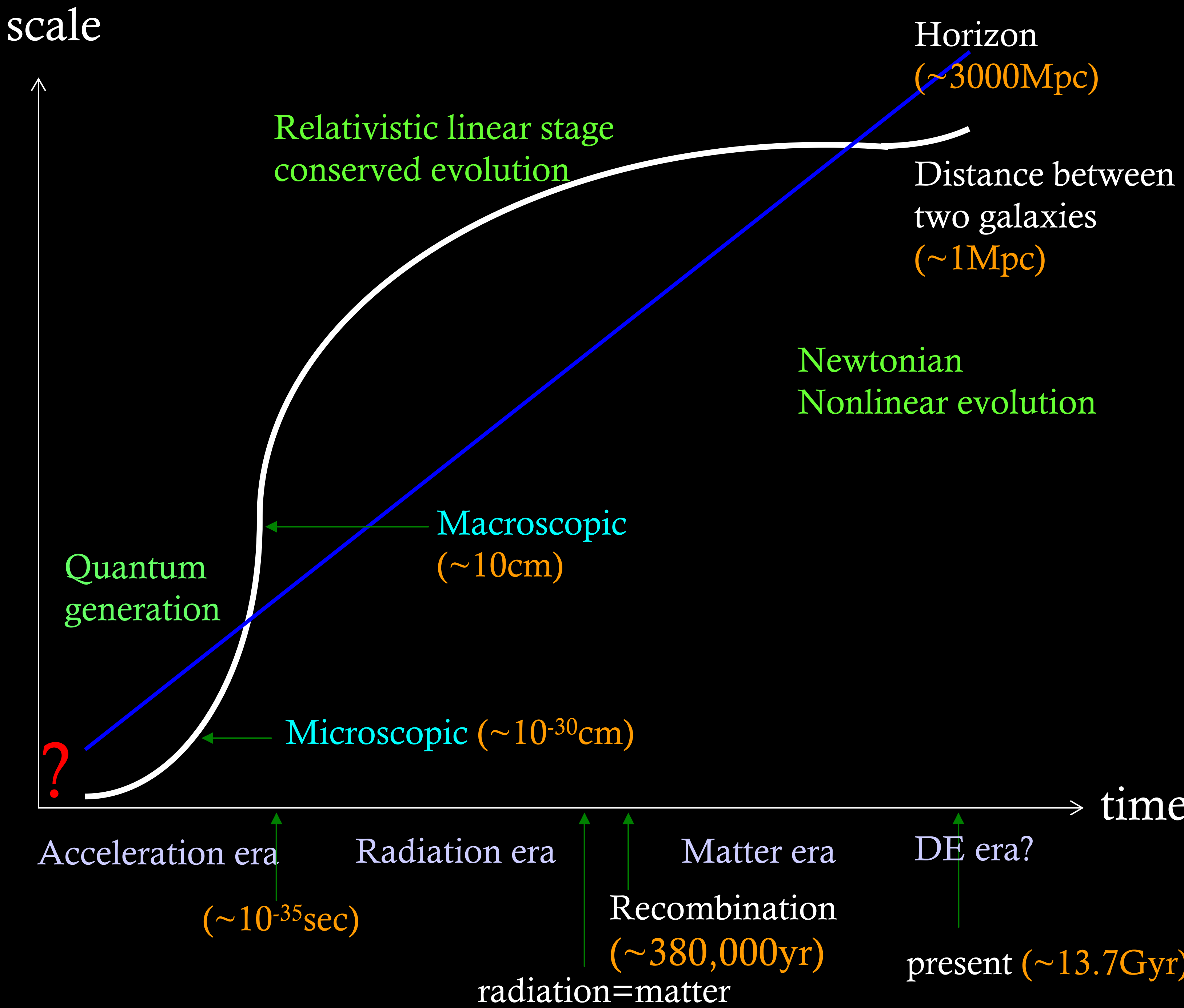
$$R^{(h)}(\mathbf{x}, t) \equiv \frac{6K(\mathbf{x}, t)}{a^2(t)} .$$

$$H = \frac{\dot{a}}{a} .$$

http://bh.knu.ac.kr/~jchan/paper/1994-ApJ-Hyun.pdf

Origin and evolution of LSS

- **Quantum origin**
  - Space-time quantum fluctuations from uncertainty pr.
  - Become macroscopic due to inflation.
- **Linear evolution (Relativistic)**
  - Linear evolution of the macroscopic seeds.
  - Structures are described by conserved amplitudes.
- **Nonlinear evolution (Newtonian)**
  - Nonlinear evolution inside the horizon.
  - Newtonian numerical computer simulation.


scale
Horizon
(~3000Mpc)
Relativistic linear stage
conserved evolution
Distance between
two galaxies
(~1Mpc)
Newtonian
Nonlinear evolution
Macroscopic
(~10cm)
Quantum
generation
Microscopic (~10^{-30}cm)
?
time
Acceleration era
Radiation era
Matter era
DE era?
(~10^{-35}sec)
Recombination
(~380,000yr)
radiation=matter
present (~13.7Gyr)

- Background world model:

  Relativistic: Friedmann (1922)

  Newtonian: Milne-McCrea (1934)

  Coincide for zero-pressure

- Linear structures:

  Relativistic: Lifshitz (1946)

  Newtonian: Bonnor (1957)

  Coincide for zero-pressure

- Second-order structures:

  Newtonian: Peebles (1980)

  Relativistic: Noh-JH (2004)

  Coincide for zero-pressure, no-rotation

- Third-order structures: Relativistic: JH-Noh (2005)

  Pure general relativistic corrections

  $\delta T/T \sim 10^{-5}$ order higher, independent of horizon

Relativistic/Newtonian correspondence:

Background order:

Spatial curvature/
Total energy

Cosmological constant

Density

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\varrho - \frac{\text{const}}{a^2} + \frac{\Lambda c^2}{3},$$

Friedmann (1922)/Milne and McCrea (1934)

Linear perturbation:

Density contrast

$$\delta(\mathbf{x}, t) \equiv \frac{\varrho(\mathbf{x}, t) - \bar{\varrho}(t)}{\bar{\varrho}(t)}.$$

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\varrho\delta = 0,$$

Lifshitz (1946)/Bonnor (1957)

http://bh.knu.ac.kr/~jchan/paper/2005-CQG-correspondence.pdf

- "It is curious that it took so long for these dynamic models to be discovered after the (more complex) general relativity models were known."

G. F. R. Ellis (1989)

- In fact, the known "Newtonian cosmology" is a version guided by Einstein's gravity!

Ellis, G. F. R., 1989, in *Einstein and the history of general relativity*, ed. D. Howard and J. Stachel (Berlin, Birkhauser), 367

Covariant equations: Ehlers (1961), Ellis (1971)

Energy conservation:

Energy frame

$$\dot{\tilde{\mu}}+(\tilde{\mu}+\cancel{\tilde{p}})\tilde{\theta}+\cancel{\tilde{\pi}^{ab}}\tilde{\sigma}_{ab}+\cancel{\tilde{q}^{a}}{}_{;a}+\cancel{\tilde{q}^{a}}\tilde{a}_{a}=0,$$

Zero-pressure

Raychaudhury equation:

$$\dot{\tilde{\theta}}+\frac{1}{3}\tilde{\theta}^{2}-\tilde{a}^{a}{}_{;a}+2(\tilde{\sigma}^{2}-\cancel{\tilde{\omega}^{2}})+4\pi G(\tilde{\mu}+3\cancel{\tilde{p}})-\Lambda=0.$$

Irrotational

http://bh.knu.ac.kr/~jchan/paper/2004-PRD-Second-order.pdf

$$\tilde{\dot{\mu}} + \tilde{\mu}\tilde{\theta} = 0, \quad \tilde{\dot{\theta}} + \frac{1}{3}\tilde{\theta}^2 + \cancel{\tilde{\sigma}^{ab}\tilde{\sigma}_{ab}} + 4\pi G\tilde{\mu} - \Lambda = 0,$$

Perturbed order

Friedmann background: $\tilde{\dot{\mu}} = \dot{\mu}, \quad \tilde{\theta} = 3\frac{\dot{a}}{a}$

$$\dot{\mu} + 3\frac{\dot{a}}{a}\mu = 0, \quad 3\frac{\ddot{a}}{a} + 4\pi G\mu - \Lambda = 0.$$

http://bh.knu.ac.kr/~jchan/paper/2006-GRG-Newtonian_vs_Relativistic.pdf

$$\dot{\tilde{\mu}} + \tilde{\mu}\tilde{\theta} = 0, \quad \dot{\tilde{\theta}} + \frac{1}{3}\tilde{\theta}^2 + \cancel{\tilde{\sigma}^{ab}\tilde{\sigma}_{ab}} + 4\pi G\tilde{\mu} - \Lambda = 0,$$

Nonlinear order

To linear-order perturbations: Comoving gauge

$$\tilde{\mu} \equiv \mu + \delta\mu, \qquad \tilde{\theta} \equiv 3\frac{\dot{a}}{a} + \delta\theta,$$

Perturbations

$$\delta\mu \equiv \delta\varrho, \qquad \delta\theta \equiv \frac{1}{a}\nabla\cdot\mathbf{u},$$

Identify

Perturbed velocity

$$\dot{\delta} + \frac{1}{a}\nabla\cdot\boldsymbol{u} = 0,$$

$$\frac{1}{a}\nabla\cdot\left(\dot{\boldsymbol{u}} + \frac{\dot{a}}{a}\boldsymbol{u}\right) + 4\pi G\mu\delta = 0.$$

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta = 0,$$

Lifshitz (1946): synchronous gauge

Nariai (1969): comoving gauge

http://bh.knu.ac.kr/~jchan/paper/2006-MN-Why_Newtonian.pdf

$$\dot{\tilde{\mu}} + \tilde{\mu}\tilde{\theta} = 0, \quad \dot{\tilde{\theta}} + \frac{1}{3}\tilde{\theta}^2 + \tilde{\sigma}^{ab}\tilde{\sigma}_{ab} + 4\pi G\tilde{\mu} - \Lambda = 0,$$

To second-order perturbations: **Comoving gauge**

$$\tilde{\mu} \equiv \mu + \delta\mu, \qquad \tilde{\theta} \equiv 3\frac{\dot{a}}{a} + \delta\theta,$$

Perturbations

$$\delta\mu \equiv \delta\varrho, \qquad \delta\theta \equiv \frac{1}{a}\nabla\cdot\mathbf{u},$$

Identify

$$\dot{\delta} + \frac{1}{a}\nabla\cdot\boldsymbol{u} = -\frac{1}{a}\nabla\cdot(\delta\boldsymbol{u}),$$

$$\frac{1}{a}\nabla\cdot\left(\dot{\boldsymbol{u}} + \frac{\dot{a}}{a}\boldsymbol{u}\right) + 4\pi G\mu\delta = -\frac{1}{a^2}\nabla\cdot(\boldsymbol{u}\cdot\nabla\boldsymbol{u})$$

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta = -\frac{1}{a^2}\frac{\partial}{\partial t}\left[a\nabla\cdot(\delta\boldsymbol{u})\right] + \frac{1}{a^2}\nabla\cdot(\boldsymbol{u}\cdot\nabla\boldsymbol{u})$$

http://bh.knu.ac.kr/~jchan/paper/2006-MN-Why_Newtonian.pdf

PHYSICAL REVIEW D **72,** 044012 (2005)

Third-order perturbations of a zero-pressure cosmological medium: Pure general relativistic nonlinear effects

Jai-chan Hwang[1] and Hyerim Noh[2]

http://bh.knu.ac.kr/~jchan/paper/2005-PRD-Third-order.pdf

Linear-order:

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta = 0,$$

Second-order:

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta = -\frac{1}{a^2}\frac{\partial}{\partial t}\left[a\nabla\cdot(\delta\mathbf{u})\right] + \frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}),$$

Third-order: **Comoving gauge**

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta = -\frac{1}{a^2}\frac{\partial}{\partial t}\left[a\nabla\cdot(\delta\mathbf{u})\right] + \frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u})$$
$$+ \frac{1}{a^2}\frac{\partial}{\partial t}\{a[2\varphi\mathbf{u} - \nabla(\Delta^{-1}X)]\cdot\nabla\delta\} - \frac{4}{a^2}\nabla\cdot\left[\varphi\left(\mathbf{u}\cdot\nabla\mathbf{u} - \frac{1}{3}\mathbf{u}\nabla\cdot\mathbf{u}\right)\right]$$
$$+ \frac{2}{3a^2}\varphi\mathbf{u}\cdot\nabla(\nabla\cdot\mathbf{u}) + \frac{\Delta}{a^2}[\mathbf{u}\cdot\nabla(\Delta^{-1}X)] - \frac{1}{a^2}\mathbf{u}\cdot\nabla X - \frac{2}{3a^2}X\nabla\cdot\mathbf{u},$$

$$X \equiv 2\varphi\nabla\cdot\mathbf{u} - \mathbf{u}\cdot\nabla\varphi + \frac{3}{2}\Delta^{-1}\nabla\cdot[\mathbf{u}\cdot\nabla(\nabla\varphi) + \mathbf{u}\Delta\varphi].$$

http://bh.knu.ac.kr/~jchan/paper/2006-GRG-Newtonian_vs_Relativistic.pdf

φ

$$ds^2 = -a^2(1+2\alpha)d\eta^2 - 2a^2\beta_{,\alpha}d\eta dx^\alpha + a^2[g^{(3)}_{\alpha\beta}(1+2\varphi) + 2\gamma_{,\alpha|\beta} + 2C^{(t)}_{\alpha\beta}]dx^\alpha dx^\beta,$$

To linear order:

Curvature perturbation

$$R^{(h)} = \frac{6\bar{K}}{a^2} - 4\frac{\Delta + 3\bar{K}}{a^2}\varphi,$$

In the comoving gauge, flat background:

$$\dot{\varphi}_v = 0. \qquad \varphi_v = C,$$

CMB:

Sachs-Wolfe effect

COBE, WMAP observations

$$\frac{\delta T}{T} \sim \frac{1}{3}\varphi_\chi = \frac{1}{3}\frac{\delta\Phi}{c^2} \sim \frac{1}{5}\varphi_v \sim \frac{1}{5}C \sim 10^{-5}$$

$$\varphi_v \sim 5 \times 10^{-5},$$

http://bh.knu.ac.kr/~jchan/paper/2005-PRD-Third-order.pdf

Mon. Not. R. Astron. Soc. **367,** 1515–1520 (2006) doi:10.1111/j.1365-2966.2006.10067.x

Why Newtonian gravity is reliable in large-scale cosmological simulations

Jai-chan Hwang[1] and Hyerim Noh[2★]

[1] *Department of Astronomy and Atmospheric Sciences, Kyungpook National University, Taegu, Korea*
[2] *Korean Astronomy and Space Science Institute, Taejon, Korea*

1. Relativistic/Newtonian correspondence to the second order.
2. Pure general relativistic third-order corrections are small $\sim 5\mathrm{x}10^{-5}$.
1. Correction terms are independent of presence of the horizon.

http://bh.knu.ac.kr/~jchan/paper/2006-MN-Why_Newtonian.pdf

Assumptions:

Our relativistic/Newtonian correspondence includes Λ, but assumes:

1. Flat Friedmann background
2. Zero-pressure
3. Irrotational
4. Single component fluid
5. No gravitational waves
6. Second order in perturbations

Relaxing any of these assumptions could lead to

pure general relativistic effects!

PHYSICAL REVIEW D **76,** 103527 (2007)

Second-order perturbations of cosmological fluids: Relativistic effects of pressure, multicomponent, curvature, and rotation

Jai-chan Hwang*

Department of Astronomy and Atmospheric Sciences, Kyungpook National University, Taegu, Korea

Hyerim Noh†

Korea Astronomy and Space Science Institute, Daejon, Korea

http://bh.knu.ac.kr/~jchan/paper/2007-PRD-Second-order-Multi.pdf

Perturbation method:

- Perturbation expansion.
- All perturbation variables are small.
- Weakly nonlinear.
- Strong gravity; fully relativistic!
- Valid in all scales!

Post-Newtonian method:

- Abandon geometric spirit of GR; recover the good old absolute space and absolute time.
- Provide GR correction terms in the Newtonian equations of motion.
- Expansion in $\frac{GM}{Rc^2} \sim \frac{v^2}{c^2} << 1$
- Fully nonlinear!
- No strong gravity situation; weakly relativistic.
- Valid far inside horizon

Cosmological nonlinear hydrodynamics with post-Newtonian corrections

Jai-chan Hwang[(a)], Hyerim Noh[(b)], and Dirk Puetzfeld[(c)]
[(a)] *Department of Astronomy and Atmospheric Sciences, Kyungpook National University, Taegu, Korea*
[(b)] *Korea Astronomy and Space Science Institute, Daejon, Korea*
[(c)] *Department of Physics and Astronomy, Iowa State University, Ames, IA, 50011, USA*

http://xxx.lanl.gov/abs/astro-ph/0507085