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Cosmological perturbations in generalised gravity theories: formulation

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Abstract. We present a simple way of deriving cosmological perturbation equations in generalised gravity theories which accounts for metric perturbations in a gauge-invariant way. We use an imperfect fluid formulation of the perturbation equations developed in Einstein gravity and absorb all new contributions as effective fluid quantities. We apply this approach to the $f(\phi, R) - \omega(\phi)\phi_{,c}\phi^{;c}$ Lagrangian which includes most of the gravity theories employing a scalar field and scalar curvature. The relation between our proposed method and the conformal transformation method is discussed. Background and perturbation equations are displayed for specific gravity theories which can be recovered as special cases from the above general Lagrangian.

1. Introduction

The linear analysis of cosmological perturbations in the Friedmann–Lemaître–Robertson–Walker (FLRW) background spacetime is important for studying the large scale structure formation process. Most of the previous work using relativistic calculations was done based on Einstein's gravity theory with the Einstein–Hilbert action. For fairly complete calculations using gauge-invariant variables and for the imperfect fluid case we can refer to Bardeen's seminal paper [1]. The same equations were derived using the covariant equations in Hwang and Vishniac† ([2], referred to as HV hereafter). HV derived all the equations in a frame which shows the contribution from the energy flux explicitly.

Recently, growing interest has arisen in modified gravity theories. These are partly motivated by quantum calculations in curved spacetime and partly by the need to construct phenomenologically successful inflationary scenarios. The first kind of motivation gives some ground for studying gravity theories modified by adding terms like R^2, $R_{ab}R^{ab}$ ($C_{abcd}C^{abcd}$ in dimensions other than four), $\xi\phi^2 R$, $R^2 \ln R$, etc to the Lagrangian [4]. Many of the models used to construct more or less successful inflationary scenarios are based on gravity theories employing some scalar fields and their direct coupling to gravity via the scalar field and scalar curvature combination. Examples

† In the covariant calculations, the Newtonian analogy of the relativistic calculations becomes apparent. To calculate the energy density perturbation, all the equations we need are the energy, momentum conservation equations and the Raychaudhuri equation, whereas in Newtonian theory we use mass conservation instead of energy conservation, and use Poisson's equation, which is basically the Raychaudhuri equation in the appropriate limit [3].

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include induced gravity inflation [5], Brans–Dicke type new inflation [6] and old inflation [7]. In the inflationary models it is well known that gravitational wave (GW) perturbations and density perturbations usually give strong constraints on the model parameters. These follow from the observed level of isotropy in the cosmic background radiation. Most inflationary models predict a minimal level of perturbations which will be generated from the ground level fluctuations of quantum fields and magnified to macroscopic size during the inflationary stage.

Although perturbation analysis based on Einstein's gravity theory is rather well studied, if we employ gravity theories which modify the basic structure of the theory, it may be necessary to redo the calculations employing the modified field equations. However, perturbation analysis in these non-Einstein gravity theories are very complicated because of the complex structure of the theories. In the present paper we will develop an efficient calculational procedure for deriving cosmological perturbation equations based on FLRW background which can treat most of the modified gravity theories within the context of the Einstein gravity theory. The basic idea is to treat all the new contributions, except the Einstein tensor part, in the field equation as contributions to the effective energy momentum tensor. Since HV recently derived all the perturbation equations in the Einstein gravity theory including imperfect fluid contributions in the particle frame, identification of the effective fluid quantities from the effective energy momentum tensor is trivial. Using simple formulae we can easily convert fluid quantities identified in the particle frame to the energy frame where no energy flux contributions appear [8]. A final simplification comes from employing the Ellis–Bruni type variables to construct the gauge-invariant (GI) perturbation variables for the density and pressure (including the entropic part) [9]. We also systematically introduced *gauge invariant and frame independent variables.*

We will apply this calculational prescription to the generalised gravitation theories based on a Lagrangian which includes the following theories as special cases; $f(R)$ gravity, which includes R^2 gravity as a special case, the most general scalar–tensor theory which includes Brans–Dicke theory as a special case, and non-minimally coupled scalar field theories which include induced gravity theory as a special case. We call this theory a '*generalised* $f(\phi, R)$ *gravity*' theory. Our calculational scheme will also be applicable to more general types of gravity theories including various fields.

It is well known that the type of theory we consider as an example contains some theories which have conformal transformation properties which can convert a given theory into Einstein's theory plus (in general) an additional scalar field with a specific potential. Many people have employed this simplifying prescription in perturbation analysis [10]. We will discuss the connection between our calculations and conformal transformation method.

In section 2, we will briefly explain the calculational procedure we propose. In section 3, the method developed in section 2 will be applied to the generalised $f(\phi, R)$ gravity theory. The full equations necessary to calculate cosmological perturbations, both for density, vorticity and gravitational wave, will be derived. In section 4 we will discuss the connections to the conformal transformation method. Section 5 is a discussion with future studies outlined. In the appendix we will display both background and density perturbation equations for specific gravity theories.

Our notation will be mostly consistent with Bardeen and HV. As in HV, a dot over a tensor quantity $T^{ab}{}_{cd}$, $\dot{T}^{ab}{}_{cd}$, means that the quantity is covariantly differentiated along the fluid 4-velocity ($\dot{T}^{ab}{}_{cd} \equiv T^{ab}{}_{cd;e}u^e$). Except in the harmonically analysed perturbation equations, we will *not distinguish between the total and perturbed quanti-*

ties using different notations. To background order it is understood that all quantities are evaluated to background order, and to linear order we only drop the higher order terms and keep the quantities as the total (see [9, 11]). We will use units where $c \equiv 1$, but our convention concerning $8\pi G$ cannot be stated at this stage, because our Lagrangian contains some theories where the resulting gravitational constant can vary in spacetime. Instead we choose $G_{ab} = T_{ab}$ to identify T_{ab}.

2. Calculational procedure

In the following section we will explain the calculational procedure for deriving the equations needed in perturbation calculations which can be applied to a rather general class of gravity theories. The method proposed in the present paper is applicable for a spacetime with FLRW background. The derived equations will be enough for single component fluid perturbations.

We may start to investigate a given theory from its Lagrangian. From the Lagrangian, using the action principle, we can derive the gravitational field equation (GFE), and the equation of motion (EOM) for the matter part. We *propose* to write the GFE in the following form:

$$G_{ab} = T_{ab} \tag{1}$$

where G_{ab} is the Einstein tensor and we have simply absorbed all the other contributions in the equations to T_{ab}. Here any possible cosmological constant (Λ) is absorbed into T_{ab} and the proportionality constant is set equal to unity. As a matter of convention we may call this T_{ab} *the effective energy momentum tensor.* We can express it in terms of effective fluid quantities as:

$$T_{ab} = \mu u_a u_b + p h_{ab} + q_a u_b + q_b u_a + \pi_{ab}$$

where μ, p, q_a, π_{ab} are the energy density, pressure, energy flux and anisotropic pressure respectively with $u_a q^a = \pi_{ab} u^b = 0$ and $\pi_{ab} = \pi_{ba}$. (For convenience, we omit the term 'effective' in referring to these fluid quantities.) The pressure can be decomposed into the equilibrium and non-equilibrium (entropic) part as $p = p_{\rm EQ} + p_{\rm NEQ}$. u^a is a 4-velocity tangent to the fluid flow lines and h_{ab} is a projection tensor into 3-space orthogonal to u^a, $h_{ab} \equiv g_{ab} + u_a u_b$. From this we can identify the fluid quantities as follows:

$$\mu = T_{ab} u^a u^b \qquad p = \tfrac{1}{3} T_{ab} h^{ab} \qquad q_a = -T_{cd} u^c h^d_a \qquad \pi_{ab} = T_{cd} h^c_a h^d_b - p h_{ab}. \tag{2}$$

The fluid quantities expressed in these equations are exact and covariant. In the case where there are no fluid fields (scalar field ϕ in our example in the next section) except the metric one, it will be convenient to use the trace part of GFE $R = -T = \mu - 3p$, as a substitute for the EOM.

Since our equation (1) is identifiable as the Einstein equation with $8\pi G \equiv 1$, we can adopt known results derived in Einstein gravity theory (e.g. Bardeen, HV). To background order, the equations we need are the following:

$$H^2 = \frac{\mu}{3} - \frac{K}{a^2} \qquad \dot{H} = -\frac{\mu + p}{2} + \frac{K}{a^2} \qquad \dot{\mu} + 3(\mu + p)H = 0 \tag{3}$$

where a is the background scale factor, K is the 3-space curvature, and H is the Hubble parameter defined as $H \equiv \dot{a}/a$. Equations (3) follow from the ADM energy constraint, the Raychaudhuri equation and the energy conservation equation in covariant equation terminology [2]. EOM can be derived from the energy conservation and the second of equations (3) can be derived from the other equations if $H \neq 0$.

For the perturbed equations, we can also use the equations derived in Bardeen and HV†. This time it is more convenient to use the set of equations displayed in section 2.1 instead of a single second-order differential equation derived for the GI density variable ϵ_m. Since most fields (including the minimally coupled scalar field) give contributions to the energy flux, as defined in equation (2), it is convenient to have perturbation equations derived in the particle frame so that we can explicitly identify the energy flux term. In this paper we adopt the notation of HV, and convert the equations derived in the particle frame to the energy frame, denoted by the superscript E, using the following property. The fluid 4-velocity in the energy frame is related to the one in the particle frame by $u^E_a = u_a + q_a/(\mu + p)$. In the harmonically analysed form we have

$$v^E = v + \frac{pf}{a(\mu + p)}. \tag{4}$$

2.1. Density perturbation

Using the transformation property in (4), we can find the following transformation rules for the GI velocity, energy density and pressure variables:

$$\begin{aligned} v_s^E &= v_s + \frac{pf}{a(\mu+p)} \qquad \mu\epsilon_m^E = \mu\epsilon_m + \frac{1}{k}\theta pf \\ \mu c_s^2 \epsilon_m^E + p\eta &= \mu c_s^2 \epsilon_m + p\eta + \frac{\dot{p}}{\dot{\mu}}\frac{\theta}{k} pf \end{aligned} \tag{5}$$

† Although we use definitions introduced in Bardeen and HV, for completeness we summarise some of the definitions below. The metric and 4-velocity are written as

$$g_{00} \equiv -a^2(1+2AY) \qquad g_{0\alpha} \equiv -a^2 BY_\alpha \qquad g_{\alpha\beta} \equiv a^2[g^{(3)}_{\alpha\beta}(1+2H_L) + 2H_T Y_{\alpha\beta}]$$

$$u_0 = -a(1+AY) \qquad u_\alpha \equiv a(v-B)Y_\alpha .$$

The Y are scalar harmonics defined as

$$Y^{|\alpha}{}_{|\alpha} = -k^2 Y \qquad Y_\alpha \equiv -\frac{1}{k}Y_{|\alpha} \qquad Y_{\alpha\beta} \equiv \frac{1}{k^2}Y_{|\alpha|\beta} + \tfrac{1}{3}g^{(3)}_{\alpha\beta}Y$$

where the bar '|' denotes a covariant differentiation based on the metric $g^{(3)}_{\alpha\beta}$. Matter variables are defined as

$$\delta\mu \equiv \mu\delta Y \qquad \delta p \equiv (c_s^2\mu\delta + p\eta)Y \qquad q_\alpha \equiv pfY_\alpha \qquad \pi^\beta_\alpha \equiv p\pi_T Y^\beta_\alpha .$$

GI variables are defined as

$$v_s \equiv v - \frac{a}{k}\dot{H}_T \qquad \epsilon_m \equiv \delta - \frac{\dot{\mu}}{\mu}\frac{a}{k}(v-B)$$

$$\Phi_H \equiv H_L + \tfrac{1}{3}H_T + \frac{\dot{a}}{k}\left(B - \frac{a}{k}\dot{H}_T\right) \qquad \Phi_A \equiv A + \frac{\dot{a}}{k}\left(B - \frac{a}{k}\dot{H}_T\right) + \frac{a}{k}\left(\dot{B} - \frac{1}{k}(a\dot{H})^{\cdot}_T\right).$$

where $\theta(\equiv u^a{}_{;a}) = 3H$ in background and $c_s^2 \equiv \mathrm{d}p/\mathrm{d}\mu$†. In the energy frame no energy flux term appears. Instead there can be a particle flux term. However, that particle flux term appears only in the number density conservation equation, and since we are only interested in the evolution of energy density, we will ignore it. So, all the equations we need expressed in the energy frame can be written as follows. (For derivations in energy and particle frame, see Bardeen and HV respectively.)

$$(\mu a^3 \epsilon_m^E)^{\cdot} = -\frac{k^2 - 3K}{k} a \left(a(\mu + p) v_s^E + \frac{2}{k} a \dot{a} p \pi_T \right) \tag{6}$$

$$\dot{v}_s^E + \frac{\dot{a}}{a} v_s^E = \frac{k}{a} \left(\Phi_A + \frac{\mu c_s^2 \epsilon_m^E + p\eta}{\mu + p} - \frac{2}{3} \frac{k^2 - 3K}{k^2} \frac{p \pi_T}{\mu + p} \right) \tag{7}$$

$$-\dot{\Phi}_H + \frac{\dot{a}}{a} \Phi_A = \frac{1}{2k} a(\mu + p) v_s^E \tag{8}$$

$$\Phi_H = \frac{a^2}{2(k^2 - 3K)} \mu \epsilon_m^E \tag{9}$$

$$\Phi_A + \Phi_H = -\frac{a^2}{k^2} p \pi_T . \tag{10}$$

Equation (6) follows from equations (8)–(10). Combining equations (7) and (8), we can derive the following:

$$\ddot{\Phi}_H + \frac{\dot{a}}{a}(3\dot{\Phi}_H - \dot{\Phi}_A) + \left(p + 2\frac{K}{a^2} \right) \Phi_A = -\tfrac{1}{2}\Delta p + \frac{k^2 - 3K}{3k^2} p \pi_T . \tag{11}$$

Here we introduced a variable Δp as $\Delta p \boxminus c_s^2 \mu \epsilon_m + p\eta + \dot{p}(a/k) v_s$. This is a GI and frame-independent variable and will be introduced later (equation (14)) in a more natural way. Equations (8)–(11) are the basic density perturbation equations derived from GFE.

Now, what we need is to substitute the fluid quantities, ϵ_m^E, v_s^E, Δp, and π_T, appearing in equations (8)–(11) in terms of the identified fluid quantities in equation (2). We will outline how to calculate the fluid quantities in a simple way. First, calculating the anisotropic pressure and the energy flux (in the particle frame) from their definitions

$$\pi_\alpha^\beta \equiv p \pi_T Y_\alpha^\beta \qquad q_\alpha = p f Y_\alpha \tag{12}$$

is trivial. Second, from equation (5) we can calculate the GI velocity variable in the energy frame, v_s^E. ($pf^E = pf(v_s^E) = 0$ also gives the same answer.) Third, the GI density variable, ϵ_m, can also be simply calculated by using the Ellis–Bruni variable as

$$h_\alpha^b \mu_{,b} (\equiv \mu \Delta_\alpha \text{ in HV}) = -k Y_\alpha \mu \epsilon_m = -k Y_\alpha \left(\Delta\mu - \frac{a}{k} \dot{\mu} v_s \right) \tag{13}$$

† In the literature, there exist inflationary models generated by allowing a bulk viscosity in the background space [12]. In our treatment of the equations, we can easily allow the background space to have an entropic (bulk viscosity) part in it. In this case we have $p = p_{\mathrm{EQ}} + p_{\mathrm{NEQ}} = \bar{p}_{\mathrm{EQ}} + \bar{p}_{\mathrm{NEQ}} + \delta p_{\mathrm{EQ}} + \delta p_{\mathrm{NEQ}}$ (an overbar indicates a background quantity) with $\delta p_{\mathrm{EQ}} \equiv c_s^2 \mu \delta Y$ and $\delta p_{\mathrm{NEQ}} \equiv p\eta_{\text{bare value}}$ where $c_s^2 \equiv \dot{p}_{\mathrm{EQ}}/\dot{\mu}$, and in (5) η is defined as

$$p\eta \equiv p\eta_{\text{bare value}} - \frac{a}{k} \dot{p}_{\mathrm{NEQ}} (v - B).$$

where $\Delta\mu (\equiv \mu\delta + (a/k)\dot{\mu}[B - (a/k)\dot{H}_T] \equiv \mu\epsilon_g$ in Bardeen) is a GI and frame-independent density variable. Using equation (5), we can derive $\mu\epsilon_m^E$. Fourth, the pressure perturbation, including the entropic part in it, can also be simply calculated using the Ellis–Bruni type variable for it:

$$h^b_\alpha p_{,b} = -kY_\alpha(\mu c_s^2\epsilon_m + p\eta) = -kY_\alpha\left(\Delta p - \frac{a}{k}\dot{p}v_s\right). \tag{14}$$

From this we can calculate Δp. The entropic perturbation is $p\eta = \Delta p - c_s^2\Delta\mu$. (In the case when the background has bulk viscosity in it, we have $\Delta p_{\rm NEQ} \equiv p\eta + (a/k)\dot{p}_{\rm NEQ}v_s = \Delta p - c_s^2\Delta\mu$.)

2.1.1. A conserved variable? In Einstein gravity many people have found a conserved quantity when the scale of the perturbation is large [13]. This variable is very convenient in connecting the perturbation spectrum at the second horizon crossing time, in the matter or radiation dominated epoch, to its spectrum at the first horizon crossing time at the inflationary stage where the perturbations are generated. This variable was generalised in [14] to general background allowing K (also Λ) and related to a conserved quantity at a sudden jump of the background equations of state. It was shown that this variable has the desired properties only when the *perfect fluid* assumption is applicable. Now, we can generalise the equations taking into account the imperfect fluid contributions. The following variable, now expressed in the energy frame, was defined in [14]

$$\psi^E \equiv -\frac{\dot{a}}{k}v_s^E + \left(1 + \frac{2}{3}\frac{k^2 - 3K}{a^2(\mu + p)}\right)\Phi_H.$$

(ψ^E becomes ζ defined in [13] for $K = 0$ and vanishing anisotropic pressure.) Using equations (7)–(10) we can show

$$\dot{\psi}^E = -\frac{k}{3a}v_s^E - \frac{\dot{a}}{a}\frac{p\eta}{\mu + p}.$$

In the large scale ($k \to 0$) limit, using equations (8) and (10), we can show

$$\dot{\psi}^E = \frac{\dot{a}}{a}\frac{1}{\mu + p}(\tfrac{2}{3}p\pi_T - p\eta).$$

So, in this large scale limit ψ^E is a conserved quantity, but only if there are negligible entropic perturbations and anisotropic pressure. In our calculation, however, since the fluid variables are defined (for calculational purposes) to absorb all the other contributions except the Einstein tensor part, there is *no a priori reason to neglect these effective imperfect fluid quantities even on very large scales*†.

† However, for the specific generalised gravity theories considered in the appendix, we recently found quantities that are conserved in the large scale for each theory and were able to express them in an unique way. For these generalisation of ζ in some GGT, see [23, 24].

2.2. Vorticity perturbation

Defining the velocity variable in the energy frame, the equations needed in calculating the vorticity perturbation evolution can be written as

$$
\begin{aligned}
[(\mu + p)a^4 v_c^E]^{\cdot} &= -\frac{k^2 - 2K}{2k} a^3 p \pi_T \\
\frac{k^2 - 2K}{2a^2}\Psi &= (\mu + p) v_c^E
\end{aligned}
\tag{15}
$$

where $\Psi \equiv v_s - v_c$†.

2.3. Gravitational wave

Gravitational wave perturbations do not depend on the frame, and the equation can be written as

$$
\ddot{H}_T + \theta \dot{H}_T + \frac{k^2 + 2K}{a^2} H_T = p\pi_T. \tag{16}
$$

This equation follows from the (α, β) component of the Einstein equation or simply from the momentum propagation equation. To calculate the anisotropic pressure, we need the following shear tensor:

$$
\sigma_{\alpha\beta} = a^2 \dot{H}_T Y_{\alpha\beta}.
$$

$Y_{\alpha\beta}$ is a tensor harmonic defined as

$$
Y_{\alpha\beta}{}^{|\gamma}{}_{|\gamma} = -k^2 Y_{\alpha\beta} \qquad Y_{\alpha\beta} = Y_{\beta\alpha} \qquad Y^{\alpha}{}_{\alpha} = Y^{\alpha\beta}{}_{|\beta} = 0.
$$

π_T and H_T are coefficients similarly defined as in scalar part but now expanded in tensor harmonics.

3. Perturbation analysis in generalised $f(\phi, R)$ gravity

In this section we will apply our proposed calculational method in the previous section

† To calculate the fluid quantities, the following kinematic properties are needed:

$$
\omega_{\alpha\beta} = a v_c Y_{[\alpha|\beta]} \qquad \sigma_{\alpha\beta} = -akv_s Y_{\alpha\beta} \qquad a_\alpha = a\left(\dot{v}_c + \frac{\dot{a}}{a} v_c\right) Y_\alpha
$$

where $v_c \equiv v - B$, $v_s \equiv v - (a/k)\dot{H}_T$. Here Y_α etc are vector harmonics defined as

$$
Y_\alpha{}^{|\beta}{}_{|\beta} = -k^2 Y_\alpha \qquad Y_{\alpha\beta} \equiv -\frac{1}{2k}(Y_{\alpha|\beta} + Y_{\beta|\alpha}) \qquad Y^{\alpha}{}_{|\alpha} = 0.
$$

v, π_T, and H_T are coefficients similarly defined as in the scalar part but now expanded in the vector harmonic.

to the following type of general Lagrangian†

$$\beta L = \tfrac{1}{2} f(\phi, R) - \tfrac{1}{2}\omega(\phi)\phi_{,c}\phi^{;c} + \beta L_{\rm M} \tag{17}$$

where β is a constant needed to fix units. (We neglect surface terms which are not relevent in this paper.)

From our general Lagrangian we can derive the following EOM for the scalar field, and GFE

$$\phi^{;c}{}_{;c} + \frac{\omega_{,\phi}}{2\omega}\phi_{,c}\phi^{;c} + \frac{f_{,\phi}}{2\omega} = 0 \tag{18}$$

$$G_{ab} = T_{ab} = \frac{1}{F}\left(\beta T^M_{ab} + \omega(\phi_{,a}\phi_{,b} - \tfrac{1}{2}g_{ab}\phi_{,c}\phi^{;c}) + g_{ab}\frac{f - RF}{2} + F_{,a;b} - g_{ab}F^{;c}{}_{;c}\right) \tag{19}$$

where we defined $F \equiv \partial f/\partial R$. If there is no scalar field ($f = f(R)$, $\omega = 0$), it is convenient to have the trace part of GFE in place of EOM

$$-R = T = (1/F)[\beta T^M - \omega\phi_{,c}\phi^{;c} + 2(f - RF) - 3F^{;c}{}_{;c}]. \tag{20}$$

Before decomposing the identified T_{ab} into fluid quantities, it is convenient to introduce the following Ellis–Bruni type variables

$$\Phi_a \equiv h^b_a\phi_{,b} \qquad R_a \equiv h^b_a R_{,b} \qquad F_a \equiv h^b_a F_{,b} \qquad f_a \equiv h^b_a f_{,b}. \tag{21}$$

These are all first-order quantities and GI because they have vanishing background values, i.e. the perturbed quantities themselves are covariant.

Using equation (2), we can derive fluid quantities as the following‡

$$\mu = \frac{1}{F}\left(\beta\mu_{\rm M} + \frac{\omega}{2}(\dot\phi^2 + \Phi_c\Phi^c) + \frac{RF - f}{2} - \theta\dot F + F^c{}_{;c} - F_a a^a\right)$$

† This Lagrangian is quite general and includes the following type of theories as special cases. (1) $f(R)$ gravity is a case with $f = f(R)$, $\omega = 0$, $\beta = 1$:

$$L = \tfrac{1}{2}f(R) + L_{\rm M}.$$

(2) R^2 gravity is a case of $f(R)$ gravity with $f(R) = R - R^2/6M^2$ [15]. (3) Generalised scalar tensor theories have $f = 2\phi[R + 2\lambda(\phi)] - 2V(\phi)$, $\beta = 16\pi$, $\omega \to 2\omega(\phi)/\phi$:

$$16\pi L = \phi(R + 2\lambda) - V - \omega\frac{\phi_{,c}\phi^{;c}}{\phi} + 16\pi L_{\rm M}.$$

These theories can be called various names depending on whether some term is considered to be a constant or neglected [16]. The most widely known case is the Brans–Dicke theory where $\lambda = V = 0$, and $\omega =$ constant. (4) The case with $f = \alpha R - \xi\phi^2 R - 2V(\phi)$, $\omega = 1$, $\beta = 1$

$$L = \tfrac{1}{2}\alpha R - \tfrac{1}{2}\xi\phi^2 R - V - \tfrac{1}{2}\phi_{,c}\phi^{;c} + L_{\rm M}.$$

$\alpha = 1$ is the non-minimally coupled scalar field case. (5) $\xi = 0$ is the minimally coupled case. (6) Induced gravity is a special case with $\alpha = 0$ with a specialised potential.

‡ In deriving these it is useful to use the following covariant identities.

$$R^{;c}{}_{;c} = -\ddot R - \theta\dot R + R^c{}_{;c} \qquad R_{,a;b}u^a u^b = \ddot R - R_a a^a$$

$$R_{,a;b}h^{ab} = -\theta\dot R + R^c{}_{;c} - R_c a^c \qquad R_{,c;d}u^c h^d_a = \dot R_a - \dot R a_a - u_a R_c a^c$$

$$R_{,c;d}h^c_a h^d_b = h^c_{(a}R_{b);c} - \tfrac{1}{3}\theta u_{(a}R_{b)} - \tfrac{1}{3}\theta\dot R h_{ab} - \dot R\sigma_{ab} - R_c u_{(a}[\omega^c{}_{b)} + \sigma^c{}_{b)}].$$

Kinematic quantities θ, σ_{ab}, ω_{ab} and a_a are defined as [17]

$$\theta \equiv u^a{}_{;a} \qquad \sigma_{ab} \equiv h^c_{(a}h^d_{b)}u_{c;d} - \tfrac{1}{3}\theta h_{ab} \qquad \omega_{ab} \equiv h^c_{[a}h^d_{b]}u_{c;d} \qquad a_a \equiv \dot u_a.$$

$$p = \frac{1}{F}[\beta p_{\rm M} + \frac{\omega}{2}(\dot\phi^2 - \tfrac{1}{3}\Phi_c\Phi^c) - \frac{RF-f}{2} + \ddot F + \tfrac{2}{3}\theta\dot F - \tfrac{2}{3}F^c{}_{;c} - \tfrac{1}{3}F_c a^c]$$

$$q_a = \frac{1}{F}[\beta q_a^M - \omega\dot\phi\Phi_a - \dot F_a + \dot F a_a + u_a F_c a^c] \tag{22}$$

$$\pi_{ab} = \frac{1}{F}[\beta\pi_{ab}^M + \omega(\Phi_a\Phi_b - \tfrac{1}{3}h_{ab}\Phi_c\Phi^c) + h^c_{(a}F_{b);c} - \tfrac{1}{3}F^c{}_{;c}h_{ab} - \dot F\sigma_{ab}$$
$$- \tfrac{1}{3}\theta u_{(a}F_{b)} - F_c u_{(a}(\omega^c{}_{b)} + \sigma^c{}_{b)}) + \tfrac{1}{3}h_{ab}F_c a^c].$$

EOM and the trace equation can also be written as the following:

$$\ddot\phi + \theta\dot\phi - \Phi^c{}_{;c} + \frac{\omega_{,\phi}}{2\omega}(\dot\phi^2 - \Phi_c\Phi^c) - \frac{f_{,\phi}}{2\omega} = 0 \tag{23}$$

$$-R = T = \frac{1}{F}[\beta T_{\rm M} + \omega(\dot\phi^2 - \Phi_c\Phi^c) + 2(f - RF) + 3(\ddot F + \theta\dot F - F^c{}_{;c})]. \tag{24}$$

All the formulae derived up to this point are covariant and exact. Since we will develop the single component fluid formulation, from now on we neglect contributions from the matter part ($T_{ab}^M \equiv 0$).

3.1. To background order

It is trivial to write the fluid quantities to background order:

$$\mu = \frac{1}{F}\left(\frac{\omega}{2}\dot\phi^2 + \frac{RF-f}{2} - \theta\dot F\right) \qquad p = \frac{1}{F}\left(\frac{\omega}{2}\dot\phi^2 + \frac{f-RF}{2} + \ddot F + \tfrac{2}{3}\theta\dot F\right)$$
$$q_a = \pi_{ab} = 0.$$

In this FLRW background, the EOM and the trace equation become

$$\begin{aligned}&\ddot\phi + \theta\dot\phi + \frac{\omega_{,\phi}}{2\omega}\dot\phi^2 - \frac{f_{,\phi}}{2\omega} = 0\\ &-R = T = \frac{1}{F}[\omega\dot\phi^2 + 2(f - RF) + 3(\ddot F + \theta\dot F)].\end{aligned} \tag{25}$$

The first two of equations (3) can be written as

$$\begin{aligned}H^2 &= \frac{1}{3F}\left(\frac{\omega}{2}\dot\phi^2 + \frac{RF-f}{2} - \theta\dot F\right) - \frac{K}{a^2}\\ \dot H &= -\frac{1}{2F}\left(\omega\dot\phi^2 + \ddot F - \tfrac{1}{3}\theta\dot F\right) + \frac{K}{a^2}.\end{aligned} \tag{26}$$

Equations (25) and (26) complete the background equations needed.

3.2. To linear order

To linear order the fluid quantities in equation (22) can be expressed as

$$
\begin{aligned}
\mu &= \frac{1}{F}\left(\frac{\omega}{2}\dot{\phi}^2 + \frac{RF-f}{2} - \theta\dot{F} + F^c{}_{;c}\right) \\
p &= \frac{1}{F}\left(\frac{\omega}{2}\dot{\phi}^2 + \frac{f-RF}{2} + \ddot{F} + \tfrac{2}{3}\theta\dot{F} - \tfrac{2}{3}F^c{}_{;c}\right) \\
q_a &= \frac{1}{F}(-\omega\dot{\phi}\Phi_a - \dot{F}_a + \dot{F}a_a) \\
\pi_{ab} &= \frac{1}{F}(F_{(a;b)} - \tfrac{1}{3}F^c{}_{;c}h_{ab} - \dot{F}\sigma_{ab}).
\end{aligned}
\tag{27}
$$

Using these in our perturbation equations (equations (8)–(11)), the EOM (equation (23)), and the trace equation (equation (24)) complete the set of equations needed to calculate cosmological density perturbations in our gravity theory.

3.2.1. Density perturbation. Following the suggested procedure we can calculate all the needed GI perturbation quantities in the energy frame, v_s^E, $\mu\epsilon_m^E$, Δp, $p\pi_T$. We will present our results in harmonically analysed forms. To do this it is convenient to introduce GI variables for ϕ, R, F, and f, in a way not depending on the frame. We already introduced the GI variables in equation (21). As an example we consider Φ_a. Since it is a spatial variable, only the spatial part does not vanish. We can expand it as $\Phi_\alpha = h^b_\alpha\phi_{,b} = -kY_\alpha[\delta\phi - (a/k)\dot{\phi}(v-B)]$. Although this is GI, it depends on choosing a frame. By decomposing the $-a\dot{\phi}v_sY_\alpha$ part we can construct a GI and frame independent variable, $\Delta\phi$,

$$
\Phi_\alpha = -kY_\alpha\left(\Delta\phi - \frac{a}{k}\dot{\phi}v_s\right) \qquad \Delta\phi \equiv \delta\phi + \frac{a}{k}\dot{\phi}\left(B - \frac{a}{k}\dot{H}_T\right). \tag{28}
$$

Similar variables can be defined for R, F and f, as ΔR, ΔF and Δf respectively. Since the shear of the normal unit vector field of the hypersurface is $\hat{\sigma}_{\alpha\beta} = -ak[B - (a/k)\dot{H}_T]Y_{\alpha\beta}$, $\Delta\phi$ (and similarly for others) measures $\delta\phi$ in the zero shear (of the normal unit vector) hypersurface [1, 2].

From equation (27) we can calculate the energy flux (in the particle frame) and anisotropic pressure as

$$
\begin{aligned}
pf &= \frac{k}{F}\left(\Delta\dot{F} - \frac{\dot{a}}{a}\Delta F + \omega\dot{\phi}\Delta\phi - \dot{F}\Phi_A\right) - a(\mu+p)v_s \\
p\pi_T &= \frac{k^2}{a^2}\frac{\Delta F}{F}.
\end{aligned}
\tag{29}
$$

Note that in these calculations, the following expansions will be used frequently:

$$
\begin{aligned}
h^b_\alpha\dot{\phi}_{,b} &= \dot{\Phi}_\alpha + \tfrac{1}{3}\theta 3\Phi_\alpha - \dot{\phi}a_\alpha = -kY_\alpha[\Delta\dot{\phi} - \dot{\phi}\Phi_A - (a/k)\ddot{\phi}v_s] \\
h^b_\alpha\ddot{\phi}_{,b} &= -kY_\alpha[\Delta\ddot{\phi} - \dot{\phi}\dot{\Phi}_A - 2\ddot{\phi}\Phi_A - (a/k)\phi^{\cdots}v_s]
\end{aligned}
$$

where $\Delta\dot{\phi} \equiv (\Delta\phi)^{\cdot}$. To calculate the fluid quantities, we will need the following kinematic quantities:

$$
\begin{aligned}
&\Theta_\alpha \equiv h^b_\alpha\theta_{,b} = -kY_\alpha[3\dot{\Phi}_H - \theta\Phi_A + (a/k)(k^2/a^2 - \dot{\theta})v_s] \\
&\sigma_{\alpha\beta} = -akv_sY_{\alpha\beta} \qquad a_\alpha = -kY_\alpha\{\Phi_A - (a/k)[\dot{v}_s + (\dot{a}/a)v_s]\}.
\end{aligned}
$$

Using equation (4) we can derive v_s^E as

$$a(\mu+p)v_s^E = \frac{k}{F}\left(\Delta\dot{F} - \frac{\dot{a}}{a}\Delta F + \omega\dot{\phi}\Delta\phi - \dot{F}\Phi_A\right). \tag{30}$$

Now using equations (13), (14) and (27) we can show the following energy density and pressure variables:

$$\mu\epsilon_m^E = \frac{1}{F}[\omega\dot{\phi}\Delta\dot{\phi} + \tfrac{1}{2}(\omega_{,\phi}\dot{\phi}^2 - f_{,\phi} + 2\omega\theta\dot{\phi})\Delta\phi + (3\dot{H} - k^2/a^2)\Delta F + \dot{F}(-3\dot{\Phi}_H + \theta\Phi_A) - \omega\dot{\phi}^2\Phi_A] \tag{31}$$

$$\Delta p = \frac{1}{F}\{\omega\dot{\phi}\Delta\dot{\phi} + \tfrac{1}{2}(\omega_{,\phi}\dot{\phi}^2 + f_{,\phi})\Delta\phi + \Delta\ddot{F} + \tfrac{2}{3}\theta\Delta\dot{F} + [\tfrac{2}{3}k^2/a^2 + \tfrac{1}{2}(p-\mu)]\Delta F + \dot{F}(2\dot{\Phi}_H - \dot{\Phi}_A) - [\omega\dot{\phi}^2 + 2(\ddot{F} + \tfrac{2}{3}\theta\dot{F})]\Phi_A\}. \tag{32}$$

With equations (29)–(32) we can rewrite our perturbation equations (8)–(11) in terms of these variables:

$$-\dot{\Phi}_H + \left(\frac{\dot{a}}{a} + \frac{\dot{F}}{2F}\right)\Phi_A = \tfrac{1}{2}\left(\frac{\Delta\dot{F}}{F} - \frac{\dot{a}}{a}\frac{\Delta F}{F} + \frac{\omega}{F}\dot{\phi}\Delta\phi\right) \tag{33}$$

$$\frac{k^2-3K}{a^2}\Phi_H + \frac{1}{2}\left(\frac{\omega}{F}\dot{\phi}^2 + \frac{3}{2}\frac{\dot{F}^2}{F^2}\right)\Phi_A = \frac{1}{2}\left\{\frac{3}{2}\frac{\dot{F}\Delta\dot{F}}{F^2} + \left(\dot{\theta} - \frac{k^2}{a^2} - \frac{3}{2}\frac{\dot{a}}{a}\frac{\dot{F}}{F}\right)\frac{\Delta F}{F} + \frac{\omega}{F}\dot{\phi}\Delta\dot{\phi} + \frac{1}{2F}\left[\omega_{,\phi}\dot{\phi}^2 - f_{,\phi} + 6\omega\dot{\phi}\left(\frac{\dot{a}}{a} + \frac{\dot{F}}{2F}\right)\right]\Delta\phi\right\} \tag{34}$$

$$\Phi_A + \Phi_H = -\Delta F/F \tag{35}$$

$$\ddot{\Phi}_H + \frac{\dot{a}}{a}\dot{\Phi}_H + \left(\frac{\dot{a}}{a} + \frac{\dot{F}}{2F}\right)(2\dot{\Phi}_H - \dot{\Phi}_A) + \left(\frac{f-RF}{2F} + \frac{2K}{a^2}\right)\Phi_A = -\frac{1}{2}\left[\frac{\Delta\ddot{F}}{F} + 2\frac{\dot{a}}{a}\frac{\Delta\dot{F}}{F} + \left(\frac{p-\mu}{2} + \frac{2K}{a^2}\right)\frac{\Delta F}{F} + \frac{\omega}{F}\dot{\phi}\Delta\dot{\phi} + \frac{1}{2F}(\omega_{,\phi}\dot{\phi}^2 + f_{,\phi})\Delta\phi\right]. \tag{36}$$

Finally operating $h_\alpha^b\partial_b$ on the EOM and trace equation (equations (23) and (24)) we have a complete set of equations needed for scalar type perturbation:

$$\Delta\ddot{\phi} + (\theta + \frac{\omega_{,\phi}}{\omega}\dot{\phi})\Delta\dot{\phi} + \left[\frac{k^2}{a^2} + \left(\frac{\omega_{,\phi}}{\omega}\right)_{,\phi}\frac{\dot{\phi}^2}{2} - \left(\frac{f_{,\phi}}{2\omega}\right)_{,\phi}\right]\Delta\phi = \dot{\phi}(\dot{\Phi}_A - 3\dot{\Phi}_H) + \Phi_A\frac{f_{,\phi}}{\omega} + \frac{f_{,\phi,R}}{2\omega}\Delta R. \tag{37}$$

$$\Delta\ddot{F} + \theta\Delta\dot{F} + (k^2/a^2 - R/3)\Delta F + \tfrac{1}{3}F\Delta R + \tfrac{2}{3}\omega\dot{\phi}\Delta\dot{\phi} + \tfrac{1}{3}(\omega_{,\phi}\dot{\phi}^2 + 2f_{,\phi})\Delta\phi = \dot{F}(\dot{\Phi}_A - 3\dot{\Phi}_H) + \tfrac{2}{3}(FR - 2f)\Phi_A. \tag{38}$$

In equation (37) we may need ΔR expressed in terms of metric quantities as follows:

$$\Delta R = 6\left[\ddot{\Phi}_H + 4\frac{\dot{a}}{a}\dot{\Phi}_H + \frac{2}{3}\frac{k^2+K}{a^2}\Phi_H - \frac{\dot{a}}{a}\dot{\Phi}_A - \left(2\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} - \frac{k^2}{3a^2}\right)\Phi_A\right].$$

Equations (33)–(38) are the (redundantly) complete equations we need for density perturbations.

3.2.2. Vorticity perturbation. We can similarly construct the GI and frame independent variable. We can show $F_\alpha = a\dot{F}v_c Y_\alpha$, and so we have $\Delta F = (a/k)\dot{F}\Psi$. From equation (27) we can derive

$$p\pi_T = \frac{k}{a}\frac{\dot{F}}{F}\Psi \qquad pf = -a(\mu + p)v_c$$

so, from equations (15) we have

$$v_c^E = 0 \quad \rightarrow \quad \Psi = 0 \quad \rightarrow \quad p\pi_T = 0.$$

That is, *the vorticity mode cannot be generated in a generalised $f(\phi, R)$ gravity theory neglecting the matter contributions.*

3.2.3. Gravitational wave perturbation. From equation (27), one can show that

$$p\pi_T = -\frac{\dot{F}}{F}\dot{H}_T$$

so, our GW perturbation equation (16) becomes

$$\ddot{H}_T + (\theta + \frac{\dot{F}}{F})\dot{H}_T + \frac{k^2 + 2K}{a^2}H_T = 0. \tag{39}$$

We emphasise that $F \equiv \partial f/\partial R \sim \partial L/\partial R$. The same equation was derived in [18] using the conformal transformation properties of the Lagrangian to the more convenient form of an Einstein type theory with scalar field. (The derivation in this method will be shown in the next section.) Ours can be considered as another derivation of the same result. For analysis of equation (39), see [18].

4. Conformal transformation method

In this section we will discuss the relation between our proposed direct method and the conformal transformation (CT) method in the case where the theory can be transformed to Einstein gravity with additional scalar field by CT. It will be shown below that our general Lagrangian can be conformally transformed into an Einstein type gravity theory with at most one additional scalar field which has a special potential [19,20].

Let us start with brief summary of the effect of CT on spacetime curvature. By CT the metric is transformed into

$$\hat{g}_{ab} = \Omega^2 g_{ab} \tag{40}$$

where Ω is a spacetime position dependent factor. (We use the hat to denote quantities based on conformally transformed metric theory.) This transformation induces the following changes in the connection coefficients and the curvature tensors

$$\hat{\Gamma}^a_{bc} = \Gamma^a_{bc} + \frac{1}{\Omega}(\Omega_{,c}\delta^a_b + \Omega_{,b}\delta^a_c - \Omega^{;a}g_{bc})$$

$$\hat{R}_{bd} = R_{bd} - \frac{1}{\Omega}(2\Omega_{,b;d} + \Omega^{;c}{}_{;c}g_{bd}) + \frac{1}{\Omega^2}(4\Omega_{,b}\Omega_{,d} - \Omega_{,c}\Omega^{;c}g_{bd})$$

$$\hat{R} = \frac{1}{\Omega^2}\left(R - \frac{6}{\Omega}\Omega^{;c}{}_{;c}\right).$$

The fluid 4-velocity and the projection tensor transform as

$$\hat{u}_a = \Omega u_a \qquad \hat{h}_{ab} = \Omega^2 h_{ab}.$$

Now, by defining the conformal factor as

$$\Omega^2 \equiv F \equiv \exp(\sqrt{\tfrac{2}{3}}\psi) \tag{41}$$

where ψ is a new dynamical variable, one can show that our original Lagrangian (equation (17)) can be transformed into (we neglect the additional matter part Lagrangian L_{M})

$$\beta\hat{L} = \tfrac{1}{2}\hat{R} - \frac{\omega}{F}\tfrac{1}{2}\phi_{,c}\phi^{\hat{;}c} - \tfrac{1}{2}\psi_{,c}\psi^{\hat{;}c} - \hat{V}(\phi,\psi) \tag{42}$$

where a possible surface term is neglected and the potential is defined as

$$\hat{V}(\phi,\psi) \equiv \frac{FR-f}{2F^2}. \tag{43}$$

It becomes obvious that our original non-Einsteinian theory is cast into an Einstein theory with (in general) an additional scalar field (ψ) with a special potential term $V(\phi,\psi)$†. From this we can derive the transformed GFE and EOM for both ϕ and ψ:

$$\hat{G}_{ab} = \hat{T}_{ab} = \frac{\omega}{F}(\phi_{,a}\phi_{,b} - \tfrac{1}{2}\phi_{,c}\phi^{\hat{;}c}\hat{g}_{ab}) + \psi_{,a}\psi_{,b} - \tfrac{1}{2}\psi_{,c}\psi^{\hat{;}c}\hat{g}_{ab} - \hat{V}\hat{g}_{ab}$$

$$\phi^{\hat{;}c}{}_{\hat{;}c} + (\omega_{,\phi}/2\omega)\phi_{,c}\phi^{\hat{;}c} - \sqrt{\tfrac{2}{3}}\psi_{,c}\phi^{\hat{;}c} - (F/\omega)\hat{V}_{,\phi} = 0$$

$$\psi^{\hat{;}c}{}_{\hat{;}c} - V_{,\psi} + \sqrt{\tfrac{2}{3}}(\omega/F)\tfrac{1}{2}\phi_{,c}\phi^{\hat{;}c} = 0.$$

The EOM for ψ corresponds to the trace equation (equation (20)) in the original theory. (We left F in these transformed equations, but it should be considered as a function of ψ defined in equation (41).)

4.1. Perturbation theory

By separating the conformal factor Ω into the background and the perturbed part as

$$\Omega \equiv \Omega_{\mathrm{BG}}(1 + \delta\Omega Y)$$

we can see that the only changes in the background scale factor and the perturbed metric are the following:

$$\hat{a} = a\Omega_{\mathrm{BG}} \qquad \hat{A} = A + \delta\Omega \qquad \hat{H}_L = H_L + \delta\Omega. \tag{44}$$

From the definitions of the potential variables one can show that changes in the GI perturbed potential (metric) variables are

$$\hat{\Phi}_A = \Phi_A + \Delta\Omega \qquad \hat{\Phi}_H = \Phi_H + \Delta\Omega \tag{45}$$

† For each of the theories shown in the appendix, the proper introduction of the new dynamical variables (Ψ) allows us to conformally transform each theory into Einstein's theory with a single MSF Ψ [24].

where $\Delta\Omega$ is a GI and frame-independent variable defined as $h^b_\alpha \Omega_{,b} = -kY_\alpha[\Omega\Delta\Omega - (a/k)\dot{\Omega}v_s]$.

In our generalised $f(\phi, R)$ gravity theory, due to equations (44) and (45), we have

$$\Omega_{\rm BG} = \sqrt{F} = \exp(\tfrac{1}{2}\sqrt{\tfrac{2}{3}}\psi) \qquad \Delta\Omega = \Delta F/2F = \tfrac{1}{2}\sqrt{\tfrac{2}{3}}\Delta\psi. \tag{46}$$

Background equations, EOM for ϕ and ψ can be written as (in the following, for notational convenience, we neglect the hat denoting quantities evaluated in $\hat{g}_{ab}$ space)

$$\begin{aligned} &H^2 = \frac{1}{3}\left(\frac{\omega}{F}\frac{\dot{\phi}^2}{2} + \frac{\dot{\psi}^2}{2} + V\right) - \frac{K}{a^2} \qquad \dot{H} = -\frac{1}{2}\left(\frac{\omega}{F}\dot{\phi}^2 + \dot{\psi}^2\right) + \frac{K}{a^2} \\ &\ddot{\phi} + \theta\dot{\phi} - \sqrt{\tfrac{2}{3}}\dot{\phi}\dot{\psi} + \frac{\omega_{,\phi}}{2\omega}\dot{\phi}^2 + \frac{F}{\omega}V_{,\phi} = 0 \\ &\ddot{\psi} + \theta\dot{\psi} + V_{,\psi} + \sqrt{\tfrac{2}{3}}\frac{\omega}{F}\tfrac{1}{2}\dot{\phi}^2 = 0. \end{aligned} \tag{47}$$

Following the procedure suggested in this paper, the perturbed fluid quantities can be derived as

$$\begin{aligned} &p\pi_T = 0 \\ &a(\mu + p)v_s^E = k(\frac{\omega}{F}\dot{\phi}\Delta\phi + \dot{\psi}\Delta\psi) \\ \mu\epsilon_m^E = &\frac{\omega}{F}\dot{\phi}\Delta\dot{\phi} + \left(\frac{\omega}{F}\theta\dot{\phi} + V_{,\phi} + \frac{\omega_{,\phi}}{2F}\dot{\phi}^2\right)\Delta\phi + \dot{\psi}\Delta\dot{\psi} + \left(\theta\dot{\psi} + V_{,\psi} - \sqrt{\tfrac{2}{3}}\frac{\omega}{F}\frac{\dot{\phi}^2}{2}\right)\Delta\psi \\ &- \left(\frac{\omega}{F}\dot{\phi}^2 + \dot{\psi}^2\right)\Phi_A \\ \Delta p = &\frac{\omega}{F}\dot{\phi}\Delta\dot{\phi} + \left(\frac{\omega_{,\phi}}{2F}\dot{\phi}^2 - V_{,\phi}\right)\Delta\phi + \dot{\psi}\Delta\dot{\psi} - \left(\sqrt{\tfrac{2}{3}}\frac{\omega}{2F}\dot{\phi}^2 + V_{,\psi}\right)\Delta\psi - \left(\frac{\omega}{F}\dot{\phi}^2 + \dot{\psi}^2\right)\Phi_A. \end{aligned}$$

Using equations (8)–(11), the perturbation equations can be written as

$$-\dot{\Phi}_H + \frac{\dot{a}}{a}\Phi_A = \frac{1}{2}\left(\frac{\omega}{F}\dot{\phi}\Delta\phi + \dot{\psi}\Delta\psi\right) \tag{48}$$

$$\begin{aligned} \frac{k^2 - 3K}{a^2}\Phi_H + \frac{1}{2}\left(\frac{\omega}{F}\dot{\phi}^2 + \dot{\psi}^2\right)\Phi_A = \frac{1}{2}\Bigg[&\dot{\psi}\Delta\dot{\psi} + \left(\theta\dot{\psi} + V_{,\psi} - \sqrt{\tfrac{2}{3}}\frac{\omega}{F}\frac{\dot{\phi}^2}{2}\right)\Delta\psi \\ &+ \frac{\omega}{F}\dot{\phi}\Delta\dot{\phi} + \left(\frac{\omega_{,\phi}}{2F}\dot{\phi}^2 + V_{,\phi} + \frac{\omega}{F}\theta\dot{\phi}\right)\Delta\phi\Bigg] \end{aligned} \tag{49}$$

$$\Phi_A + \Phi_H = 0 \tag{50}$$

$$\begin{aligned} \ddot{\Phi}_H + \frac{\dot{a}}{a}(3\dot{\Phi}_H - \dot{\Phi}_A) + \left(-V + \frac{2K}{a^2}\right)\Phi_A = -\frac{1}{2}\Bigg[&\dot{\psi}\Delta\dot{\psi} - \left(V_{,\psi} + \sqrt{\tfrac{2}{3}}\frac{\omega}{2F}\dot{\phi}^2\right)\Delta\psi \\ &+ \frac{\omega}{F}\dot{\phi}\Delta\dot{\phi} + \left(\frac{\omega_{,\phi}}{2F}\dot{\phi}^2 - V_{,\phi}\right)\Delta\phi\Bigg]. \end{aligned} \tag{51}$$

EOM for ϕ and ψ are

$$\Delta\ddot{\phi} + \left(\theta - \sqrt{\tfrac{2}{3}}\dot{\psi} + \frac{\omega_{,\phi}}{\omega}\dot{\phi}\right)\Delta\dot{\phi} + \left[\frac{k^2}{a^2} + \left(\frac{\omega_{,\phi}}{2\omega}\right)_{,\phi}\dot{\phi}^2 + F\left(\frac{V_{,\phi}}{\omega}\right)_{,\phi}\right]\Delta\phi$$
$$- \sqrt{\tfrac{2}{3}}\dot{\phi}\Delta\dot{\psi} + \frac{(V_{,\phi}F)_{,\psi}}{\omega}\Delta\psi = \dot{\phi}(\dot{\Phi}_A - 3\dot{\Phi}_H) - 2\frac{F}{\omega}V_{,\phi}\Phi_A \qquad (52)$$

$$\Delta\ddot{\psi} + \theta\Delta\dot{\psi} + \left(\frac{k^2}{a^2} + V_{,\psi\psi} - \frac{1}{3}\frac{\omega}{F}\dot{\phi}^2\right)\Delta\psi + \sqrt{\frac{2}{3}}\frac{\omega}{F}\dot{\phi}\Delta\dot{\phi} + \left(V_{,\psi,\phi} + \sqrt{\frac{2}{3}}\frac{\omega_{,\phi}}{2F}\dot{\phi}^2\right)\Delta\phi$$
$$= \dot{\psi}(\dot{\Phi}_A - 3\dot{\Phi}_H) - 2V_{,\psi}\Phi_A. \qquad (53)$$

Using equations (44)–(46) one can trivially check that the original set of equations for the background and perturbation (equations (25), (26), (33)–(38)) can be derived from the above conformally transformed set of the equations. In fact, *this can be another way of deriving the perturbation equations in these generalised gravity theories using* CT *properties.* For vanishing additional scalar field ($\phi = 0$), the above equations are the case with *minimally coupled scalar field* ψ *with special potential* $V(\psi)$.

The vorticity mode cannot be generated in this type of theory neglecting the matter part. Since $p\pi_T = 0$ in the GW mode, the GW equation becomes

$$H_T'' + 2\frac{\hat{a}'}{\hat{a}}H_T' + (k^2 + 2K)H_T = 0$$

where a prime denotes a derivative with respect to conformal time. Using equation (44) one can easily derive the GW perturbation equation in our original theory expressed in equation (39). For discussions about the physical significance of two conformally related metrics see [21].

5. Discussion

In the present paper we have presented a simple way of deriving linear perturbation equations which can be applicable to a broad range of gravity theories, by treating the system in analogy with Einstein's equation. We absorbed all new contributions into the effective energy momentum tensor and treated them as fluid like contributions. Derivations of these equations become simpler using the Ellis–Bruni type variables and introducing the gauge-invariant and frame-independent variables. The formalism developed here is applicable to the single component fluid case in the FLRW background. Multi-component fluid generalisation is straightforward and will be presented elsewhere.

Although we have restricted our attention to a generalised $f(\phi, R)$ gravity theory, this prescription can be applied to wider variety of gravity theories, e.g., allowing the $R_{ab}R^{ab}$ term in the Lagrangian, $H^{(3)}_{ab}$ contributions to T_{ab} [4], etc. Applications to specific gravity theories will be of great interest in view of the recent growing interest in these theories. In particular, the generation and evolution of perturbations during a possible inflationary phase and their final spectrum at second horizon crossing time deserve special attention. All these questions are currently under investigation. (See [23, 24] for recent advances.)

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Appendix. Equations for specific gravity theories

Here we will display the background and perturbed equations needed in the perturbation calculation in specialised gravity theories. These equations can be derived in three different ways. First, we can directly apply the method proposed in this paper. Second, we can reduce the equations derived for generalised $f(\phi, R)$ gravity. Third, we can use the CT method and transform back to the original theory. Analysis of these equations in known background evolution, especially including inflationary epochs, will be presented elsewhere†.

A1. f(R) gravity

$f(R)$ gravity is a case with $f = f(R)$, $\omega = 0$, $\beta = 1$.
Background:

$$\mu = \frac{1}{F}\left(-\theta\dot{F} + \frac{RF - f}{2}\right) \qquad p = \frac{1}{F}\left(\ddot{F} + \tfrac{2}{3}\theta\dot{F} - \frac{RF - f}{2}\right)$$
$$\ddot{F} + \theta\dot{F} + \tfrac{1}{3}(2f - FR) = 0.$$

Perturbed equations:

$$-\dot{\Phi}_H + \left(\frac{\dot{a}}{a} + \frac{\dot{F}}{2F}\right)\Phi_A = \frac{1}{2}\left(\frac{\Delta\dot{F}}{F} - \frac{\dot{a}}{a}\frac{\Delta F}{F}\right)$$

$$\frac{k^2 - 3K}{a^2}\Phi_H + \frac{3}{4}\frac{\dot{F}^2}{F^2}\Phi_A = \frac{1}{2}\left[\frac{3}{2}\frac{\dot{F}\Delta\dot{F}}{F^2} + \left(\dot{\theta} - \frac{k^2}{a^2} - \frac{3}{2}\frac{\dot{a}}{a}\frac{\dot{F}}{F}\right)\frac{\Delta F}{F}\right]$$

$$\Phi_A + \Phi_H = -\Delta F/F$$

$$\ddot{\Phi}_H + \frac{\dot{a}}{a}\dot{\Phi}_H + \left(\frac{\dot{a}}{a} + \frac{\dot{F}}{2F}\right)(2\dot{\Phi}_H - \dot{\Phi}_A) + \left(\frac{f - RF}{2F} + \frac{2K}{a^2}\right)\Phi_A$$
$$= -\frac{1}{2}\left[\frac{\Delta\ddot{F}}{F} + 2\frac{\dot{a}}{a}\frac{\Delta\dot{F}}{F} + \left(\frac{p-\mu}{2} + \frac{2K}{a^2}\right)\frac{\Delta F}{F}\right]$$

$$\Delta\ddot{F} + \theta\Delta\dot{F} + \left(\frac{k^2}{a^2} - \frac{R}{3}\right)\Delta F + \tfrac{1}{3}F\Delta R = \dot{F}(\dot{\Phi}_A - 3\dot{\Phi}_H) + \tfrac{2}{3}(FR - 2f)\Phi_A.$$

† Recently we were able to rewrite the equations displayed in each of the following gravity theories, and found asymptotic solutions in both the large and small scale limits [23]. The large scale asymptotic solutions and corresponding conserved quantities can be used to discuss the general process of calculating the inflationary spectrum in a generic inflationary model [24].

A2. R^2 gravity

R^2 gravity is a case of $f(R)$ gravity with $f(R) = R - R^2/6M^2$.
Background:

$$\mu = \frac{1}{3M^2F}\left(\theta\dot{R} - \frac{1}{4}R^2\right) \qquad p = \frac{1}{3M^2F}\left(-\ddot{R} - \tfrac{2}{3}\theta\dot{R} + \frac{1}{4}R^2\right)$$

$$\ddot{R} + \theta\dot{R} - M^2R = 0.$$

Perturbed equations:

$$-\dot{\Phi}_H + \left(\frac{\dot{a}}{a} - \frac{\dot{R}}{6M^2F}\right)\Phi_A = \frac{1}{6M^2F}\left(-\Delta\dot{R} + \frac{\dot{a}}{a}\Delta R\right)$$

$$\frac{k^2 - 3K}{a^2}\Phi_H + \frac{\dot{R}^2}{12M^4F^2}\Phi_A = \frac{1}{6M^2F}\left[\frac{\dot{R}\Delta\dot{R}}{2M^2F} + \left(\frac{k^2}{a^2} - \dot{\theta} - \frac{\dot{a}}{a}\frac{\dot{R}}{2M^2F}\right)\Delta R\right]$$

$$\Phi_A + \Phi_H = \frac{\Delta R}{3M^2F}$$

$$\ddot{\Phi}_H + \frac{\dot{a}}{a}\dot{\Phi}_H + \left(\frac{\dot{a}}{a} - \frac{\dot{R}}{6M^2F}\right)(2\dot{\Phi}_H - \dot{\Phi}_A) + \left(\frac{R^2}{12M^2F} + \frac{2K}{a^2}\right)\Phi_A$$
$$= \frac{1}{6M^2F}\left[\Delta\ddot{R} + 2\frac{\dot{a}}{a}\Delta\dot{R} + \left(\frac{p-\mu}{2} + \frac{2K}{a^2}\right)\Delta R\right]$$

$$\Delta\ddot{R} + \theta\Delta\dot{R} + \left(\frac{k^2}{a^2} - M^2\right)\Delta R = \dot{R}(\dot{\Phi}_A - 3\dot{\Phi}_H) + 2M^2R\Phi_A.$$

For $K = 0$, these equations are derived in [15].

A3. Generalised scalar tensor theory

Generalised scalar tensor theories is a case with $f = 2\phi[R + 2\lambda(\phi)] - 2V(\phi)$, $\beta = 16\pi$, $\omega \to 2\omega(\phi)/\phi$ (since λ can be absorbed into V, we neglect λ in the following).
Background:

$$\mu = \frac{V}{2\phi} + \frac{\omega}{2}\frac{\dot{\phi}^2}{\phi^2} - \theta\frac{\dot{\phi}}{\phi} \qquad p = -\frac{V}{2\phi} + \frac{\omega}{2}\frac{\dot{\phi}^2}{\phi^2} + \frac{1}{\phi}\left(\ddot{\phi} + \tfrac{2}{3}\theta\dot{\phi}\right)$$

$$\left(1 + \frac{3}{2\omega}\right)(\ddot{\phi} + \theta\dot{\phi}) = -\frac{\omega_{,\phi}}{2\omega}\dot{\phi}^2 + \frac{1}{\omega}\left(V - \frac{\phi}{2}V_{,\phi}\right).$$

Perturbed equations:

$$-\dot{\Phi}_H + \left(\frac{\dot{a}}{a} + \frac{\dot{\phi}}{2\phi}\right)\Phi_A = \frac{1}{2}\left[\frac{\Delta\dot{\phi}}{\phi} + \left(\omega\frac{\dot{\phi}}{\phi} - \frac{\dot{a}}{a}\right)\frac{\Delta\phi}{\phi}\right]$$

$$\frac{k^2 - 3K}{a^2}\Phi_H + \tfrac{1}{2}(\omega + \tfrac{3}{2})\frac{\dot{\phi}^2}{\phi^2}\Phi_A = \frac{1}{2}\left\{(\omega + \tfrac{3}{2})\frac{\dot{\phi}\Delta\dot{\phi}}{\phi^2} + \left[\left(\frac{\omega}{\phi}\right)_{,\phi}\frac{\dot{\phi}^2}{2} + \frac{V_{,\phi}}{2} - \frac{V}{\phi}\right.\right.$$

$$-\frac{k^2-3K}{a^2}+\frac{\omega}{2}\frac{\dot\phi^2}{\phi^2}+(\omega+\tfrac{3}{2})\theta\frac{\dot\phi}{\phi}\Big]\frac{\Delta\phi}{\phi}\Big\}$$

$$\Phi_A+\Phi_H=-\Delta\phi/\phi$$

$$\ddot\Phi_H+\frac{\dot a}{a}\dot\Phi_H+\left(\frac{\dot a}{a}+\frac{\dot\phi}{2\phi}\right)(2\dot\Phi_H-\dot\Phi_A)+\left(-\frac{V}{2\phi}+\frac{2K}{a^2}\right)\Phi_A$$
$$=-\frac{1}{2}\left\{\frac{\Delta\ddot\phi}{\phi}+\left(\omega\frac{\dot\phi}{\phi}+2\frac{\dot a}{a}\right)\frac{\Delta\dot\phi}{\phi}+\left[\left(\frac{\omega}{\phi}\right)_{,\phi}\frac{\dot\phi^2}{2}+\frac{2K}{a^2}-\frac{V_{,\phi}}{2}-p\right]\frac{\Delta\phi}{\phi}\right\}$$

$$\Delta\ddot\phi+\left[\theta+\left(\ln\frac{\omega}{\phi}\right)^{\cdot}\right]\Delta\dot\phi+\left[\frac{k^2}{a^2}+\left(\ln\frac{\omega}{\phi}\right)_{,\phi\phi}\frac{\dot\phi^2}{2}-\frac{1}{2}\left(\frac{\phi}{\omega}\right)_{,\phi}R+\frac{1}{2}\left(\frac{\phi}{\omega}V_{,\phi}\right)_{,\phi}\right]\Delta\phi$$
$$=\dot\phi(\dot\Phi_A-3\dot\Phi_H)+\frac{\phi}{\omega}(R-V_{,\phi})\Phi_A+\frac{\phi}{2\omega}\Delta R.$$

A4. Non-minimally coupled scalar field and induced gravity

$f=\alpha R-\xi\phi^2R-2V(\phi)$, $\omega=1$, $\beta=1$, $\alpha=1$ in the non-minimally coupled case ($\xi=0$ in minimally coupled case). Induced gravity is a special case with $\alpha=0$ with a specialised potential.

Background:

$$\mu=\frac{1}{\alpha-\xi\phi^2}\left(\frac{\dot\phi^2}{2}+V+2\xi\theta\phi\dot\phi\right)\qquad p=\frac{1}{\alpha-\xi\phi^2}\left[\frac{\dot\phi^2}{2}-V-2\xi\phi\left(\ddot\phi+\tfrac{2}{3}\theta\dot\phi+\frac{\dot\phi^2}{\phi}\right)\right]$$

$$\ddot\phi+\theta\dot\phi+\xi\phi R+V_{,\phi}=0.$$

Perturbed equations:

$$-\dot\Phi_H+\left(\frac{\dot a}{a}+\frac{-\xi\phi\dot\phi}{\alpha-\xi\phi^2}\right)\Phi_A=\frac{1}{\alpha-\xi\phi^2}\left\{-\xi\phi\Delta\dot\phi+\left[\frac{\dot\phi}{2}+\xi\phi\left(\frac{\dot a}{a}-\frac{\dot\phi}{\phi}\right)\right]\Delta\phi\right\}$$

$$\frac{k^2-3K}{a^2}\Phi_H+\frac{1}{\alpha-\xi\phi^2}\left(\frac{1}{2}+\frac{3\xi^2\phi^2}{\alpha-\xi\phi^2}\right)\dot\phi^2\Phi_A=\frac{1}{\alpha-\xi\phi^2}\left\{\left(\frac{1}{2}+\frac{3\xi^2\phi^2}{\alpha-\xi\phi^2}\right)\dot\phi\Delta\dot\phi\right.$$
$$\left.+\left[-\ddot\phi\left(\frac{1}{2}+\frac{3\xi^2\phi^2}{\alpha-\xi\phi^2}\right)+\xi\phi\frac{k^2-3K}{a^2}\right]\Delta\phi\right\}$$

$$\Phi_A+\Phi_H=\frac{2\xi\phi\Delta\phi}{\alpha-\xi\phi^2}$$

$$\ddot\Phi_H+\frac{\dot a}{a}\dot\Phi_H+\left(\frac{\dot a}{a}-\frac{\xi\phi\dot\phi}{\alpha-\xi\phi^2}\right)(2\dot\Phi_H-\dot\Phi_A)+\left(-\frac{V}{\alpha-\xi\phi^2}+\frac{2K}{a^2}\right)\Phi_A$$
$$=-\frac{1}{\alpha-\xi\phi^2}\left\{-\xi\phi\Delta\ddot\phi+\left[\frac{\dot\phi}{2}-2\xi\left(\dot\phi+\frac{\dot a}{a}\phi\right)\right]\Delta\dot\phi\right.$$
$$\left.+\left[-\frac{V_{,\phi}}{2}+\xi\phi\left(p-\frac{\ddot\phi}{\phi}-2\frac{\dot a}{a}\frac{\dot\phi}{\phi}-\frac{2K}{a^2}\right)\right]\Delta\phi\right\}$$

$$\Delta\ddot\phi+\theta\Delta\dot\phi+\left(\xi R+V_{,\phi\phi}+\frac{k^2}{a^2}\right)\Delta\phi=\dot\phi(\dot\Phi_A-3\dot\Phi_H)-2(V_{,\phi}+\xi\phi R)\Phi_A-\xi\phi\Delta R.$$

A5. Minimally coupled scalar field

Background:

$$\mu = \frac{\dot{\phi}^2}{2} + V \qquad p = \frac{\dot{\phi}^2}{2} - V \qquad \ddot{\phi} + \theta\dot{\phi} + V_{,\phi} = 0.$$

Perturbation equations:

$$- \dot{\Phi}_H + \frac{\dot{a}}{a}\Phi_A = \tfrac{1}{2}\dot{\phi}\Delta\phi$$

$$\frac{k^2 - 3K}{a^2}\Phi_H + \tfrac{1}{2}\dot{\phi}^2\Phi_A = \tfrac{1}{2}(\dot{\phi}\Delta\dot{\phi} - \ddot{\phi}\Delta\phi)$$

$$\Phi_A + \Phi_H = 0$$

$$\ddot{\Phi}_H + \frac{\dot{a}}{a}(3\dot{\Phi}_H - \dot{\Phi}_A) + \left(-V + \frac{2K}{a^2}\right)\Phi_A = -\tfrac{1}{2}(\dot{\phi}\Delta\dot{\phi} - V_{,\phi}\Delta\phi)$$

$$\Delta\ddot{\phi} + \theta\Delta\dot{\phi} + \left(V_{,\phi\phi} + \frac{k^2}{a^2}\right)\Delta\phi = \dot{\phi}(\dot{\Phi}_A - 3\dot{\Phi}_H) - 2V_{,\phi}\Phi_A.$$

For $K = 0$, these equations are derived in [22].

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