EVOLUTION OF IDEAL-FLUID COSMOLOGICAL PERTURBATIONS

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ABSTRACT

We present solutions of cosmological perturbations for an ideal-fluid medium with a constant pressure \( p \) to density \( \rho \) relation in a flat Friedmann-Lemaître-Robertson-Walker (FLRW) background model. Solutions are presented for six different gauge choices which include most of those conventionally used. Both exact and asymptotic solutions are shown in tabular forms. Solutions span a range in \( w \equiv p/\rho \) from \(-1\) to 1. On subhorizon scales except for the uniform-density gauge, for general \( w \), the density perturbations in all gauge choices behave in the same way; also for a pressureless medium the behavior is in accord with the Newtonian results. In the large scale (generally meaning scales where the pressure gradient term is negligible, thus corresponding to superhorizon scales for a medium with vanishing pressure, and to general scales for a pressureless medium), the integral-form solutions are known which can cover changing \( w \) in time as long as the ideal-fluid assumption is valid; i.e., for general \( \rho \). Using this we show that on superhorizon scales, except for the zero-shear gauge and the uniform-curvature gauge, the growing mode of the perturbed curvature variable \( \varphi \) remains constant considering general \( \rho \), and the dominating decaying mode cancels out. A central role will be played by an integration constant denoted \( C \) which contains information about spatial structures and of course remains constant. Knowing \( C \) we can determine the growing mode part of all the variables. On superhorizon scales \( C \) can be identified with \( \varphi \)'s in many different gauge choices. We also identify the perturbed density variable in the comoving gauge and the perturbed velocity and potential variables in the zero-shear gauge which have correct correspondences with the Newtonian perturbed density, velocity, and potential variables, respectively. We note that in our approach, if the solution of a variable is known in a given gauge, all the rest of the solutions in all the other gauges can be simply derived unless the gauge is ill-behaved. Using the full solutions as a guide, attempts have been made to clarify relations between solutions known in the literature.

Subject headings: cosmology: theory — galaxies: formation

1. INTRODUCTION

Relativistic linear perturbation analysis in a homogeneous and isotropic medium is a well-developed subject. The analysis was originally presented in a classic study by Lifshitz (1946) assuming a particular gauge called the synchronous gauge in which the gauge freedom was not completely fixed. The remaining gauge mode is not a big problem for the theory, as long as the spurious mode is carefully traced as was done in Lifshitz’s analysis. However, the existence of the gauge mode caused generations of confusion and also generated a belief that the subject is full of pitfalls. However, in most cases, the problems were caused by pure algebraic errors probably due to the internal complexity of the general relativistic system. As in electromagnetic theory or in the more general gauge theories, the gauge freedom can be used as an advantage rather than as an obstacle in treating a given problem. This point was made in a paper by Bardeen (1988), and extended in Hwang (1991c).

In particular, the gauge nonspecific method and the equations suggested in Bardeen (1988) can be optimally used in relating results between different gauge-specific and also gauge-invariant methods. The method is simply to write the complete set of equations without specifying any gauge. Due to the spatial homogeneity of the background, the equations can be written in terms of the variables independent of the spatial gauge transformations. Thus, adapting the system to a specific gauge choice, which may depend on the character of each given problem, becomes trivial. In this way one can postpone choosing the gauge until later, and thus one is able to adopt a gauge after deciding which choice is most suitable for a given problem. Also, since the translation into other gauges becomes trivial, in many cases this way of approaching the problem may provide some advantage. We recall that each set of gauge-invariant (GI) variables and the method based on the GI variables do in fact correspond to a specific gauge choice: “the GI variables and equations are closely related to the variables and perturbation equations in certain specific gauges” (Bardeen 1980). Specifically, in the six different fundamental gauge choices to be presented, with the exception of the synchronous gauge, all of the other gauge choices completely fix the gauge mode which is thus in itself free of gauge mode. These can also be easily written in the corresponding GI forms. In fact, most of the GI formulations used in the literature are based on the comoving, or zero-shear GI variables.

In this paper we use the equations in Bardeen (1988) and its algebraic extensions made in Hwang (1991c; H1 hereafter), and derive the complete solutions (both exact and asymptotic) for an ideal fluid with a constant pressure to density ratio in a flat FLRW background. We also present the large-scale integral form of solutions covering the temporally changing background equation of state. Although self-contained, this paper may be considered as an addendum to § 3.4 of H1.

In § 2 we summarize our notation and equations. We also introduce the six different fundamental gauge choices which encompass most of the conventionally used gauges and describe the relation to explicitly GI methods. In § 3 we describe the method of solving the equations in a particular gauge and also describe how to translate the result into all of the other gauges. Then, using this method we present both the exact and asymptotic solutions in the ideal-fluid medium spanning \( w \) from \(-1\) to 1. Also presented are the large-scale
integral-form solutions valid for general \( p(\mu) \). Correspondences of our relativistic analysis with the Newtonian results are made in §§ 2.4 and 3.4.1. Section 4 is a summary. Although the solutions presented in tabular form are the main results of this paper, the equations and the methods employed to relate the solutions between different gauges are more generally applicable to other problems.

We let \( K = 0 = \Lambda \) in most cases for mathematical simplicity; \( K \) is the sign of the background three-space curvature and \( \Lambda \) is the cosmological constant. However, as long as \( w > -1/3 \) (this may include ordinary stages), the terms containing \( K \) and \( \Lambda \) decay more slowly compared to the density term in equation (6). Thus, if we consider that at the present stage the terms are at best comparable, as we go earlier the \( \mu \) term becomes more important compared to the \( K \) and \( \Lambda \) terms. In this case one can assume \( K = 0 = \Lambda \) without worry. However, for evolution near the current epoch, assuming \( w = 0 \), the exact solutions including both \( K \) and \( \Lambda \) can be found in § 3.7 (eq. [67]).

Considering the cases with general \(-1 \leq w \leq 1\) may go beyond pure pedagogic interest. For example, in most cosmological models, especially those considering early stages of evolution, it is a common practice to assume a power-law expansion law \( a \propto t^\alpha \), where \( a(t) \) is a cosmic scale factor. This case simply corresponds to assuming \( n = 2/(3 + 3w) \) (eq. [7]); thus to have an \( n > 1 \) stage, which is desirable to ensure that the currently observable scales were once in causal contact, we need \( w < -1/3 \) (Bardeen 1980, § VII). Although most of the solutions are derived for constant \( w \) stages, the integral-form solutions given in § 3.7 (eq. [63]) allow us to cover changing \( w \) on superhorizon scales as long as we have \( p = p(\mu) \). This result following from dynamic evolution will be incorporated in our superhorizon scale discussion in § 3.3. We will show that on superhorizon scales, independent of changing \( w \) (thus \( n \)), there exists a conserved variable in most of the gauge choices (§§ 3.3.2 and 3.8).

The criteria for large scale and small scale depends on whether the “spatial pressure gradient terms” appearing in the wave equations are negligible or dominating compared to the other gravity terms present (e.g., see eq. [25]). For a medium with nonvanishing pressure the dividing scale \((\lambda \sim c \tau)\) becomes comparable to the visual horizon \((\sim c t)\) or Hubble horizon \((\sim c H^{-1})\); in this case we have in mind the medium with ordinary pressure like radiation-dominated era (RDE) where \( c_t = c/s^{1/2} \). In the following the horizon means visual or Hubble horizon. In matter-dominated era (MDE) the criteria becomes Jeans length \((\sim c \tau)\) which is negligible compared to the horizon \((c_t, < c)\); thus in MDE small scale is not relevant, at least in this paper. Even in MDE the horizon scale appears as a dividing scale for behavior of some variables. By superhorizon or subhorizon scales we mean on scales larger or smaller than the horizon, respectively; still the criteria depends on whether the spatial gradient term (besides the pressure gradient term in MDE case) is negligible or dominating, respectively. For a medium with nonvanishing pressure, the large scale (small scale) can be considered as superhorizon (subhorizon) scales.

In this paper we set \( c = 1 \).

2. EQUATIONS

2.1. Notation

Following Bardeen (1988), we introduce the perturbed metric as

\[
\begin{align*}
g_{00} &= -a^2(1 + 2\alpha) , \\
g_{0a} &= -a^2 \beta_a , \\
g_{ab} &= a^2 [g^{(3)}_{ab}(1 + 2\rho) + 2\gamma_{ab}] ,
\end{align*}
\]

where \( g^{(3)}_{ab} \) is a comoving three-space part of the FLRW metric; a vertical bar denotes a covariant derivative based on the metric \( g^{(3)}_{ab} \); indices \( \alpha, \beta, \ldots \) are space indices. In this paper we consider only a scalar-type mode representing the evolution of density condensation and of corresponding matter and metric quantities. In FLRW space other (vector and tensor type) modes are completely decoupled. Since the background we are considering is homogeneous in space, one can make the equations invariant under spatial gauge transformations (Bardeen 1988). Since \( \beta \) and \( \gamma \) depend on a spatial gauge transformation (eq. [A5]) we introduce a combination of these variables \( \chi \) which does not

\[
\chi \equiv a(\beta + a\gamma) , \quad \kappa \equiv 3(H\alpha - \dot{\phi}) + \frac{k^2}{a^2} \chi ,
\]

where \( k^2 \) is a comoving wave number introduced as \( \chi^{(s)}_{ab} \equiv \nabla^{(s)} \chi \equiv -k^2 \chi \).

In a normal frame where the frame vector has a property \( n_a = 0 \) one can show

\[
R^{(3)} = \frac{6K}{a^3} + 4 \frac{k^2 - 3K}{a^2} \phi , \quad \theta = 3H - \kappa ,
\]

where \( R^{(3)}, \theta, \) and \( \sigma_{ab} \) are three-space curvature normal to \( n_a \), expansion, and shear of the frame vector \( n_a (a, b, \ldots \) are space-time indices). From these follow that \( \alpha, \phi, \kappa, \) and \( \chi \) may be interpreted as the perturbations in the lapse function, the three-space curvature, the expansion, and the shear, respectively. From equation (3) the amplitude of the shear in the normal frame follows

\[
\sigma \equiv \frac{1}{\sqrt{2}} \sqrt{\sigma_{ab} \sigma_{ab}} = \frac{k^2}{a^2} \chi \sqrt{\frac{1}{2} Y^{ab} Y_{ab}} .
\]

The variable \( Y_{ab} \) is spatial harmonic functions based on \( g^{(3)}_{ab} \) (see eq. [1] of H1). The matter perturbations are introduced as

\[
\delta \mu \equiv \epsilon , \quad \delta \rho \equiv \pi \equiv c_t^2 \epsilon + e ,
\]

where \( \mu, \rho, \) and \( \pi_{ab} \) are the energy-density, pressure, energy-flux, and anisotropic pressure, respectively; \( c_t^2 = \delta / \mu \). We introduce a four-velocity \( u^a \), and since \( u^a \equiv u^a + q^a / (\mu + p) \) is frame-independent, \( \Psi \) defined as \( q_a + (\mu + p) n_a \equiv \Psi_n \) is a frame-independent variable indicating fluid four-velocity in the energy frame \( (q_a = 0) \); thus in the energy frame we have \( u^a \equiv u^a \) which is why we introduced an index \( E \) and energy flux in the normal frame \( (n_a = n_a) \) where \( n_a = 0 \); see H1 and Hwang (1991d; H2 hereafter).

2.2. Equations

2.2.1. Background

The background equations are

\[
H^2 = \frac{8\pi G}{3} \frac{\mu - K}{a^2} + \frac{\Lambda}{3} , \quad \dot{H} = -4\pi G(\mu + p) + \frac{K}{a^2} ,
\]

which follow from the Arnowitt-Deser-Misner (ADM) energy constraint and the Raychaudhuri equations, respectively. In this paper we often consider \( K = 0 = \Lambda \) and \( w \equiv p/ \mu \).
\[ \mu = \text{constant case where equations (6) can be solved as} \]
\[ a \propto t^{2(1 + 3\omega)} \propto \eta^{2(1 + 3\omega)}, \]
\[ (7) \]
where \( \eta \) is a conformal time defined as \( d\eta = a^{-1} \, dt \).

2.2.2. Perturbations

In the following we will present the complete equations we need in perturbation analysis in FLRW background. For derivations based on the ADM and the covariant equations, we refer to Bardeen (1988) and H1; compared to H1 we recovered 8\( \pi G \):

\[ \dot{\phi} = H\phi - \frac{1}{3} \kappa + \frac{1}{3} \frac{k^2}{a^2} \chi, \]
\[ (8) \]
\[ - \frac{k^2 - 3K}{a^2} \varphi + H\kappa = -4\pi G\epsilon, \]
\[ (9) \]
\[ \kappa - \frac{k^2 - 3K}{a^2} \chi = -12\pi G\Psi, \]
\[ (10) \]
\[ \dot{\chi} + H\chi - a - \varphi = 8\pi G\sigma, \]
\[ (11) \]
\[ \dot{\kappa} + 2H\kappa = \left( \frac{k^2}{a^2} - 3H \right) \chi + 4\pi G(\epsilon + 3\kappa), \]
\[ (12) \]
\[ \dot{\epsilon} + 3H(\epsilon + \pi) = (\mu + p)(\kappa - 3H\alpha) + \frac{k^2}{a^2} \Psi, \]
\[ (13) \]
\[ \dot{\Psi} + 3H\Psi = - (\mu + p)\alpha - \pi + \frac{2}{3} \frac{k^2 - 3K}{a^2} \sigma. \]
\[ (14) \]

These equations are valid including \( \Lambda \).

2.3. Gauges

Under the gauge transformation \( \tilde{x} = x^\alpha + \zeta^\alpha \), our perturbation variables transform as (see § 2.2 of H1)

\[ \tilde{x} = x - \tilde{T}, \quad \tilde{\phi} = \varphi - HT, \]
\[ \tilde{\chi} = \chi - T, \quad \tilde{\kappa} = \kappa + \left( \frac{3H - \frac{k^2}{a^2}}{a^2} \right) T, \]
\[ \tilde{\epsilon} = \epsilon - \tilde{\mu} T, \quad \tilde{\pi} = \pi - \tilde{p} T, \quad \tilde{\Psi} = \Psi + (\mu + p) T, \]
\[ \tilde{\sigma} = \sigma, \quad \tilde{\chi} = \epsilon, \quad \tilde{\delta} = \delta \phi - \phi \tilde{T}, \]
\[ (15) \]

where \( \phi(t) \) is any scalar function with \( \delta \phi \) as a perturbation; \( T \equiv a^2 \). Thus, fixing any one of the following variables, \( x, \varphi, \chi, \kappa, \epsilon, \pi, \text{or} \Psi \) can be used as a gauge fixing condition. We may denote these as follows:

- synchronous gauge (SG), \( \kappa \equiv 0 \);
- comoving gauge (CG), \( \Psi \equiv 0 \);
- zero-shear gauge (ZSG), \( \chi \equiv 0 \);
- uniform-expansion gauge (UEG), \( \kappa \equiv 0 \);
- uniform-curvature gauge (UCG), \( \varphi \equiv 0 \);
- uniform-density gauge (UDG), \( \epsilon \equiv 0 \).

In the literature, the CG is sometimes called the Lagrangian gauge (Sakai 1969); the ZSG is often called the longitudinal, or conformal Newtonian gauge (Kodama & Sasaki 1984 and Mukhanov, Feldman, & Brandenberger 1992); the UEG was called the uniform Hubble-constant gauge (Bardeen 1980); the UDG was once called the pure metric-fluctuation system (Schön 1989). We can call \( \pi = 0 \) a uniform-pressure gauge, but in an ideal fluid this case can be absorbed into the UDG.

Note that in the SG, fixing \( x = 0 \) (so that the proper time interval becomes the same as the background coordinate-time interval), \( T \) is still not completely determined up to an arbitrary constant. With the exception of this gauge, all the other gauge conditions completely fix the gauge mode (see eq. [15]). This implies that except for the SG we can construct corresponding GI formulations. We can also note that in equations (8)–(14) \( \hat{z} \) does not appear which lead the equations in the SG to be third-order whereas in all the other gauges it is a second-order differential equation. In some problems, especially gravitational waves and the post-Newtonian approximations, the de Donder (also called harmonic) gauge is conveniently used (Weinberg 1973). This gauge has also been used in the cosmological perturbation analysis in the literature (Irvine 1965; Arons & Silk 1968; Schön 1989). This gauge can be considered as a combination, \( \hat{z} + H\chi + \kappa = 0 \), of our fundamental gauges, and the equation results in a fourth-order differential equation in which two gauge modes need to be removed and is thus probably not particularly illuminating. In the Appendix we summarize the de Donder gauge in a cosmological context.

H1 introduced a simple way of denoting GI variables as follows: From equation (15) one can easily see that the following variables are GI:

\[ \epsilon_\Psi \equiv \epsilon - 3H\Psi, \quad \Psi_\chi = \Psi + (\mu + p)\chi, \quad \varphi_\chi \equiv \varphi - H\chi, \quad \sigma_\chi = \sigma - \hat{z}, \]
\[ (16) \]
and so on. In this way, for example, \( \epsilon_\Psi \) uniquely specify a GI variable made from \( \epsilon \) and \( \Psi \) which becomes \( \epsilon \) in the CG; equivalently it becomes \( -3H\Psi \) in the UDG. Some exceptions arise in the variables involving \( x \) as an index; e.g., \( \epsilon_x \equiv \epsilon - \tilde{\mu} \int x \, dt \) is not GI since the lower bound of the integration gives rise to a gauge mode. In the SG by taking \( x = 0 \) there still remains a gauge mode causing \( t^{-1} \) temporal behavior in \( \tilde{z} = \epsilon_\mu / \mu \). Thus in any gauge-specific method, except for the SG, by changing every perturbation variable into the corresponding GI variable we can arrive at the corresponding GI equations; e.g., in the CG by simply letting \( \epsilon \rightarrow \epsilon_\Psi, \chi \rightarrow \chi_\Psi, \) etc., one can arrive at a GI set of equations which is in fact closely related to the CG equations. One can check that our fundamental equations (eqs. [8]–[14]) are gauge-independent and can be expressed in terms of various GI combinations (§ 3.2 of H1). Commonly used GI methods in the literature usually employ some combination of the CG and the ZSG variables:

\[ \epsilon_\mu Y = \epsilon_\Psi \equiv \delta_\Psi, \quad v_\mu Y = -\frac{k}{a} \frac{\Psi_\chi}{\mu + p}, \]
\[ \Phi_\mu Y = \Phi_\chi, \quad \Phi_\mu Y = \chi_\mu, \quad \delta_\mu \chi = \delta_\chi. \]
\[ (17) \]

As mentioned in Bardeen (1980), in this expression it is apparent that \( \varphi_\chi \) and \( \sigma_\chi \) measure the amount of curvature warping and the amplitude of the fractional perturbation in the lapse function of the zero-shear hypersurface. Other variables can also be similarly interpreted; Bardeen (1980) used the subindices \( m, g \) and \( h \) which correspond to our subindices \( \Psi, \chi \), and \( k \), respectively.

Bardeen (1988) introduced the following variable

\[ \zeta \equiv \varphi + \frac{1}{3} \frac{\epsilon}{\mu + p} = \varphi_\chi, \]
\[ (18) \]
(see eq. [7.1] of Bardeen 1980; Bardeen, Steinhardt, & Turner 1983). The importance of this variable as a conservation variable on superhorizon scales and characterizing adiabatic per-
perturbation is noted in Bardeen (1988) and H2. In fact, we will see that \( \varphi \) in many gauge choices (one of them is \( \zeta \)) are conserved on superhorizon scales; see equations (41), (73), and (74).

### 2.4. Newtonian Correspondences

The variables \( \epsilon \) and \( \psi \) may have direct physical significance. A variable \( \epsilon_\psi \equiv \sigma_{u_b} Y \) is identified in Hwang & Vishniac (1990; HV1 hereafter) as the spatial gradient of density distribution orthogonal to the fluid flow (\( h_{\rho,\mu}^b \)) introduced in Olson (1976) and Ellis & Bruni (1989); the variable \( h_{\rho,\mu}^b \) is both covariant and GI (based on the CG). The variable \( \epsilon_\psi \) also has a direct physical interpretation in terms of the shear of the matter velocity field (Bardeen 1980; see also eq. [33] of HV1); i.e., we have (eqs. [114], [33] of HV1),

\[
h_{\rho,\mu}^b = \mu \epsilon_{\mu} Y_{,\mu}, \quad \sigma_{\mu b}(u_b) = -a_k v_{,b} Y_{,b}.
\]

(19)

In our frame-independent definitions, both \( \sigma_{\mu b} \) and \( h_{\rho,\mu}^b \) in equation (19) are based on \( \epsilon_{\mu}^b \); whereas \( \sigma_{\mu b} \) in equation (3) was based on \( u_{,\mu} = n_{,\mu} \) where \( n_{,\mu} = 0 \). The amplitude of shear of \( \epsilon_{\mu}^b \) flow behaves as

\[
\sigma_{\mu b}(u_b) \equiv \frac{1}{\sqrt{2}} \frac{\sigma^{ab} \sigma_{ab}}{a} \frac{1}{\sqrt{2}} \frac{1}{Y^* P_{,b}} Y_{,b},
\]

(20)

From equations (9) and (10) one can derive the following

\[
k^2 - 3K \quad \frac{\varphi}{a^2} = 4\pi G \epsilon_\psi.
\]

(21)

This equation resembles Poisson’s equation in the Newtonian theory and will be conveniently used later in connecting the ZSG variable to the CG variable; perturbed Newtonian gravitational potential may go as \( k^2 \delta \Phi = -(k^2 - 3K) \varphi \). From this together with additional arguments made in § 3.4.1 and Hwang & Hyun (1993; HH1 hereafter) we may conclude: “\( \varphi_{,\mu} \) and \( \epsilon_{\psi} \) do correspond to the Newtonian perturbed potential and density, respectively”, and similarly, “\( \psi_{,\mu} \) corresponds to the Newtonian perturbed velocity.”

Taking the UEG (\( \kappa = 0 \)), equation (9) directly gives the Poisson’s equation relating \( \varphi \) and \( \epsilon \) as

\[
k^2 - 3K \quad \frac{\varphi}{a^2} = 4\pi G \epsilon.
\]

(22)

In MDE and on subhorizon scales both \( \epsilon, \varphi, \) and \( \Psi (\sim \psi) \) in the UEG reproduce correct Newtonian results. Whereas, for a case with general pressure (including \( p = 0 \)) on superhorizon scales the behavior of all these variables in the UEG deviate from the corresponding behavior of \( \epsilon, \varphi, \) and \( \Psi, \) respectively. These can be checked with our complete solutions to be presented later. For more details, see HH1.

### 3. METHODS AND SOLUTIONS WITH COMMENTS

#### 3.1. A Method of Translations

If we have a solution, the rest of solutions in the same gauge can be derived using equations (8)–(14). If we have solutions in a gauge, we can use these to derive solutions in the other gauges. As an example to be used in the following calculation, let us derive \( \epsilon \) in the UEG using known solutions in the CG. From equation (15) one can see that

\[
\epsilon_{,\mu} - \epsilon_{,\psi} = \left( 3 \frac{\dot{H}}{H} - \frac{k^2}{a^2} \right)^{-1} \kappa + 3H \Psi.
\]

(23)

Imposing the UEG (thus let \( \kappa = 0 \)) we have \( \epsilon = \epsilon_{\psi} + 3H \Psi \). Using equation (10) and the definition of \( \chi_{\psi} \) [from eq. (16) we have \( \chi_{\psi} \equiv \chi + \Psi (\mu + p) \)] one can express \( \Psi \) in terms of \( \chi_{\psi} \) which gives

\[
\epsilon = \epsilon_{\psi} + 3H \left( \frac{1}{\mu + p} + \frac{12\pi G a^2}{k^2 - 3K} \right)^{-1} \chi_{\psi}.
\]

(24)

Using equations (8)–(14), from \( \epsilon \) one can derive the rest of the variables in the UEG.

We may summarize a way of deriving variables for each gauge choice starting from the CG (or corresponding GI) solutions. In the CG we have \( \epsilon = \epsilon_{\psi} \). In the ZSG we have \( \varphi = \varphi_{,\psi} \) which is directly related to \( \epsilon_{\psi} \) through equation (21). In the UCG we can use \( \varphi_{,\psi} = -H \psi \). In the UDGI we can use \( \epsilon_{\psi} = -3H \psi \). In the SG, equation (11) implies \( \dot{\psi} = \varphi_{,\psi} \) so that \( \psi = \int \varphi_{,\psi} \, dt \) where the lower bound of integration gives rise to the gauge mode. The case of the UEG has been explained above (eq. [24]). From a known solution the rest follow from equations (8)–(14).

#### 3.2. Exact Solutions

We may start from a well-known solution in the CG as shown in Bardeen (1980; see also Sakai 1969). For completeness we derive the solution from our fundamental equations (eqs. [8]–[14]). Assuming the CG, for an ideal fluid (\( \pi = c_s^2 \epsilon \)) with \( K = 0 \) and \( \pi = p/\mu \) constant, equations (12)–(14) can be combined to give

\[
d + (2 - 3w)Hd + \left[ c_s^2 \frac{k^2}{a^2} - 4\pi G (1 - w)(1 + 3w) \right] \delta = 0,
\]

(25)

where we used \( \dot{w} = 3H(1 + w)(1 - c_s^2) \). Thus, for constant \( w \), \( c_s^2 = w \). (From eqs. [13], [14] one can see that when \( w \) is exactly equal to \(-1\), the CG does not behave properly, due to cancellation of terms with \( \mu + p \). This case will be treated separately later in § 3.6.)

Defining

\[
f \equiv x^2 \delta, \quad x \equiv kH \beta \frac{k}{aH}, \quad \beta \equiv \frac{2}{1 + 3w},
\]

(26)

(thus \( a \propto \eta^3 \)) we can rewrite equation (25) in terms of the spherical Bessel function (Bardeen 1980, eq. [4.5])

\[
x'' + 2x' + [w^2 - \beta (\beta + 1)]f = 0,
\]

(27)

where a prime denotes a derivative with respect to \( x \). (At present we exclude the case with \( w = -1/3 \) where \( \beta \) diverges. This case will be treated separately later in § 3.5.)

Solutions can be written in terms of the spherical Bessel functions (Sakai 1969 and Bardeen 1980)

\[
f = \tilde{a} j_{\frac{x}{\sqrt{w}}} + \tilde{b} n_{\frac{x}{\sqrt{w}}}.
\]

(28)

This form is not convenient for the \( w = 0 \) case where the exact solution simply becomes

\[
f = cx^2 + dx^3
\]

(29)

This solution applies on scales larger than Jeans length in MDE. As will be noted later, this minor point was a source of some confusion in the literature; see § 3.4. The RDE where \( w = 1/3 \) (\( \beta = 1 \)) is also an especially interesting case where the solution is well known in terms of the spherical Bessel functions of first order:

\[
j_s(x) = \frac{\sin y}{y^2} - \frac{\cos y}{y}, \quad n_s(x) = -\frac{\cos y}{y^2} - \frac{\sin y}{y}.
\]

(30)
The exact solutions for each variable in the six different gauges are shown in Table 1. We present these solutions in implicit form; however, writing out these solutions in explicit form is trivial (use eq. [28] or eq. [29]). Some solutions in the SG have been explicitly integrated and expressed in terms of Lommel functions in Ratra (1988). Some variables are expressed in dimensionless form; e.g., κ/H (eq. [3]) and δ (eq. [5]). Considering equation (4) we present (x/β)^2 H_x which is of order σ(n)/H. Although the shear of the u_x flow behaves as (see

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Ψ = 0 (Comoving)</th>
<th>χ = 0 (Zero Shear)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\gamma\Psi/H)</td>
<td>0</td>
<td>(3\beta x^{-\delta-1} f)</td>
</tr>
<tr>
<td>(\left(\frac{x}{\beta}\right)^2 H_x)</td>
<td>(\frac{3}{2(1 + \beta)} x^{-\delta-3/2} f)</td>
<td>0</td>
</tr>
<tr>
<td>κ/H</td>
<td>(-\frac{3}{2(1 + \beta)} x^{-\delta-3/2} f)</td>
<td>(-\frac{3}{4} \beta x^{-\delta-1} f)</td>
</tr>
<tr>
<td>(\alpha)</td>
<td>(-\frac{2 - \beta}{2(1 + \beta)} x^{-\delta-3/2} f)</td>
<td>(-\frac{3}{4} \beta x^{-\delta-1} f)</td>
</tr>
<tr>
<td>(\varphi)</td>
<td>(\frac{3}{2} \beta^2 x^{-\delta-1} f + \frac{3\beta^2}{2(1 + \beta)} x^{-\delta-1} f)</td>
<td>(\frac{3}{4} \beta^2 x^{-\delta} f)</td>
</tr>
<tr>
<td>(\delta)</td>
<td>(x^{-\delta-3/2} f)</td>
<td>(x^{-\delta-3/2} f + 3\beta x^{-\delta-1} f)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameter</th>
<th>κ = 0 (Uniform Expansion)</th>
<th>α = 0 (Synchronous)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\gamma\Psi/H)</td>
<td>(A^{-1} \frac{1}{1 + \beta} x^{-\delta-3/2} f)</td>
<td>(3\beta x^{-\delta-1} f - \beta^2(1 + \beta)x^{-\delta-1} f)</td>
</tr>
<tr>
<td>(\left(\frac{x}{\beta}\right)^2 H_x)</td>
<td>(A^{-1} \frac{3}{2(1 + \beta)} x^{-\delta-3/2} f)</td>
<td>(\frac{3}{4} \beta x^{-\delta-1} f)</td>
</tr>
<tr>
<td>κ/H</td>
<td>0</td>
<td>(-\frac{3}{4} \beta x^{-\delta-1} f + \frac{3}{4} \beta^2(1 + \beta)A^{-1} x^{-\delta-1} f)</td>
</tr>
<tr>
<td>(\alpha)</td>
<td>(-A^{-1} \frac{1}{1 + \beta} x^{-\delta-3/2} f - A^{-2} \frac{1}{(1 + \beta)^2} x^{-\delta-3/2} f)</td>
<td>0</td>
</tr>
<tr>
<td>(\varphi)</td>
<td>(\frac{3}{2} \beta^2 x^{-\delta-1} f + A^{-1} \frac{3\beta^2}{2(1 + \beta)} x^{-\delta-1} f)</td>
<td>(\frac{3}{4} \beta^2 x^{-\delta} f + \frac{3}{4} \beta^2 x^{-\delta-1} f)</td>
</tr>
<tr>
<td>(\delta)</td>
<td>(x^{-\delta-3/2} f + A^{-1} \frac{1}{1 + \beta} x^{-\delta-3/2} f)</td>
<td>(x^{-\delta-3/2} f + 3\beta x^{-\delta-1} f - 3\beta^2(1 + \beta)x^{-\delta-1} f)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameter</th>
<th>(\varphi = 0) (Uniform Curvature)</th>
<th>(\epsilon = 0) (Uniform Density)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\gamma\Psi/H)</td>
<td>(3\beta(1 + \beta)x^{-\delta-1} f + 3\beta x^{-\delta-1} f)</td>
<td>(-x^{-\delta-1} f)</td>
</tr>
<tr>
<td>(\left(\frac{x}{\beta}\right)^2 H_x)</td>
<td>(-\frac{1}{2} x^{-\delta-3/2} f)</td>
<td>(\frac{1}{2\beta(1 + \beta)} x^{-\delta+1} f + \frac{3}{2(1 + \beta)} x^{-\delta-3/2} f)</td>
</tr>
<tr>
<td>κ/H</td>
<td>(-\frac{3}{2} \beta(1 + \beta)A^{-1} x^{-\delta-1} f - \frac{3}{4} \beta x^{-\delta-1} f)</td>
<td>(\frac{3}{2} A^{-1} x^{-\delta+1} f + \frac{3}{2(1 + \beta)} x^{-\delta-3/2} f)</td>
</tr>
<tr>
<td>(\alpha)</td>
<td>(-\frac{3}{2} \beta(1 + \beta) x^{-\delta-1} f - \frac{3}{4} \beta x^{-\delta-1} f)</td>
<td>(\frac{1}{2} x^{-\delta+1} f + \frac{1}{2(1 + \beta)} x^{-\delta-3/2} f)</td>
</tr>
<tr>
<td>(\varphi)</td>
<td>0</td>
<td>(\frac{3}{2} \beta^2 A^{-1} x^{-\delta-1} f + \frac{3\beta^2}{2(1 + \beta)} x^{-\delta-1} f)</td>
</tr>
<tr>
<td>(\delta)</td>
<td>(3\beta(1 + \beta)A^{-1} x^{-\delta-1} f + 3\beta x^{-\delta-1} f)</td>
<td>0</td>
</tr>
</tbody>
</table>

Note.—The exact solutions are presented in implicit form which are representative and concise. We assume a single component ideal fluid with \(w = p/\mu = \text{constant}\) and \(K = 0 = A\). \(\delta_0 = x^{-\delta-1/2} f\), etc., where \(\beta = 2(1 + 3\omega)\) and \(f\) can be expressed in terms of Bessel functions (eq. [28]): \(f' = df/dx; A = 1 + x^2/3[\beta(1 + \beta)]; x = k\phi \propto \epsilon^{1/3} \Delta \propto \lambda^{1/3} \Delta; \gamma = 8\pi G\). The lower bound of integration in the SG causes the presence of the gauge mode in that gauge. This tables does not properly include the case with \(w = -1 (\beta = -1)\) and \(w = -\frac{3}{2} (\beta\) diverges). These cases are considered in Tables 7 and 6, respectively.
IDEAL-FLUID COSMOLOGICAL PERTURBATIONS

\[ \frac{\sigma(u_k)}{H} \approx \frac{k}{aH} v_s = -\frac{4\pi G x^2}{\beta(1 + \beta)} H \Psi_x - \frac{4\pi G x^2}{\beta(1 + \beta)} H \Psi \beta^2 x^2 H_x, \]

we present \( \Psi \) as a \( 8\pi G \Psi / H \) combination. The variable in equation (31) was defined as \( \zeta \) in Bardeen (1980).

The variable \( f \) is related to the GI variables identified in § 2.4 as

\[ \delta = x^{-\beta + 2} f, \quad \varphi_x = \frac{3}{2} \beta^2 x^{-\beta} f, \]

\[ v_s Y = -\frac{k \Psi_x}{a \mu + p} = -\frac{3}{2} \beta x^{-\beta + 1} f. \]

Since \( \zeta \) can be also written as

\[ \zeta = \varphi_x + \frac{1}{\mu + p} \left( H \Psi_x + \frac{1}{3} \epsilon_w \right), \]

we have

\[ \zeta = \frac{3}{2} \beta \frac{1}{1 + \beta} x^{-\beta} \left[ (1 + \beta) + \frac{1}{3} \beta x^2 \right] f + x f', \]

For the shear of \( u_x^2 \) flow we have

\[ \frac{\sigma(u_k)}{H} \approx \frac{k}{aH} v_s = -\frac{3}{2(1 + \beta)} x^{-\beta + 2} f. \]

3.3. Superhorizon Scale Evolution

On superhorizon scales (\( c_s x \ll 1 \); in this section we exclude the pressureless case which will be generally treated in § 3.4, thus for medium with nonvanishing \( w \) the criteria can be considered as the horizon scale), for \( \beta \) a non-half-integer, equation (28) can be expanded as

\[ f = c \left[ x^\beta + \frac{\beta - 2}{6\beta(2\beta + 1)} x^{\beta + 2} + \ldots \right] \]

\[ + \frac{d}{2(2\beta - 1)(2\beta - 3)} x^{-\beta + 3} + \ldots, \]

where

\[ c = \frac{\beta^2}{2^{\beta - 1}} \sqrt{\Gamma(\beta + 1)} \left( \Gamma(2\beta + 2) \right), \]

\[ d = b(-1)^{\beta + 1} \cos(\beta n) \frac{\Gamma(\beta)}{2^{\beta - 1} \Gamma^{\beta + 1/2} \Gamma(\beta)}. \]

(For \( \beta \) a half-integer, one can use the solution expressed in terms of the Neumann function.) Due to the cancellation of the dominant decaying mode of \( \epsilon \) in the SG, we have displayed the decaying mode to third-order in the expansion (this cancellation in the SG occurs if \( w \neq 0 \); for \( w = 0 \) one should use eq. [29] instead and in this case the cancellation of the dominant decaying mode does not generally occur; see § 3.4). Although the solutions are presented in Table 1 of H1, in the following

we present it again now using \( C \) where

\[ C = e^{\frac{3\beta^2(2\beta + 1)}{2(\beta + 1)}}. \]

Later, after solving the dynamic equations involving general \( p(\mu) \) (thus including changing \( \beta \)) we will show that \( C \) remains constant even when the background equation of state (and thus \( w \) or \( \beta \)) changes (see eqs. [39], [63], [65]). In § 3.7 we will show that \( C \) is in fact an integration constant for the integral-form solutions generally available for every solution in the large-scale considering general \( p(\mu) \). The complete solutions valid on superhorizon scales are presented in Table 2.

In the SG there remain gauge modes. For each variable the gauge mode is proportional to \( x^{\beta - 1} \propto t^{-1} \) on any scale \( (\kappa H \) is an exception, see Table 1). The behavior of the gauge mode can be read directly from Table 1 where the gauge mode arises from the lower bound of the integration of \( f \) \( f \) \( dx \) terms. Some of the decaying modes of the SG in Table 2 are not valid in a \( w = 0 \) (thus \( \beta = 2 \)) medium; e.g., the cancellation occurring in the decaying mode of \( \epsilon \) in the SG does not occur in the \( w = 0 \) case. (The \( w = 0 \) case will be treated separately in § 3.4 using eq. [29] and Table 1.) Some divergences occurring in the SG for \( \beta = -3, 4 \) are caused from the improper integration of \( x^{-1} \) and the correct coefficient for these specific cases is given in Table 3.

The solutions in Table 3 are derived for the non-half-integer \( \beta \) case. For the half-integer \( \beta \) case we should use the expansion of \( f \) in terms of the Neumann function. Differences in this case occur only when \( \beta = 1/2 \) which caused divergences in many of the decaying modes in Table 3. Thus we have presented the solutions for \( \beta = 1/2 \) (\( w = 1 \), thus corresponding to ultra-relativistic stiff fluid) separately in Table 4. Notice that the only differences appear in the coefficient of the decaying mode and the temporal behavior remains the same (except for some additional logarithmic dependences).

3.3.1. Comments on Gauge-invariant Variables

The behavior of some of the GI variables evolve as

\[ \delta = \frac{2(\beta + 1)}{3\beta^2(2\beta + 1)} x^2 + dx x^{-2\beta + 1}, \]

\[ v_s Y = -\frac{1}{2\beta + 1} x + d \frac{3}{2} \beta x^{-2\beta}, \]

\[ \varphi_x = \frac{\beta + 1}{2\beta + 1} + d \frac{3}{2} \beta^2 x^{-2\beta - 1}, \]

\[ \zeta = C + d \frac{\beta}{2(2\beta - 1)} x^{-2\beta + 1}. \]

Although the decaying mode does not vanish in an exact way, the dominating decaying mode has cancelled in \( \zeta \). The growing mode part of \( \zeta \) remains constant. On superhorizon scales, from the dynamic equations we can show that \( \zeta \) remains constant independent of the changes in the background equation of states (eq. [73]). The coefficient \( C \) as defined in equation (38) is a \( \beta \)-independent constant. For the shear of \( u_x^2 \) flow we have

\[ \frac{\sigma(u_k)}{H} \approx \frac{k}{aH} v_s Y = -\frac{C x^2}{2(2\beta + 1)} + d \frac{3}{2} x^{-2\beta + 1}. \]
### Table 2

**Solutions Valid on Superhorizon Scales**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\Psi = 0$ (Comoving)</th>
<th>$\chi = 0$ (Zero Shear)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma$</td>
<td>$0$</td>
<td>$C \frac{2(\beta + 1)}{2\beta + 1} - \frac{3(\beta + 1)}{2\beta + 1} x^{-2\beta - 1}$</td>
</tr>
<tr>
<td>$\frac{x^2}{\beta} H_{X}$</td>
<td>$C \frac{1}{\beta(2\beta + 1)} x^2 - d \frac{3}{2} x^{-2\beta - 1}$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\kappa/H$</td>
<td>$C \frac{1}{\beta(2\beta + 1)} x^2 - d \frac{3}{2} x^{-2\beta - 1}$</td>
<td>$-C \frac{3(\beta + 1)}{2\beta + 1} + \frac{3}{2} \beta \frac{g}{\beta + 1} x^{-2\beta - 1}$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$C \frac{\beta - 2}{3\beta^2(2\beta + 1)} x^2 + d \frac{\beta - 2}{2\beta + 1} x^{-2\beta - 1}$</td>
<td>$-C \frac{\beta + 1}{2\beta + 1} - d \frac{3}{2} \beta x^{-2\beta - 1}$</td>
</tr>
<tr>
<td>$\varphi$</td>
<td>$C - d \frac{\beta}{2(\beta - 1)(\beta + 1)} x^{-2\beta - 1}$</td>
<td>$C \frac{\beta + 1}{2\beta + 1} + d \frac{3}{2} \beta x^{-2\beta - 1}$</td>
</tr>
<tr>
<td>$\delta$</td>
<td>$C \frac{2(\beta + 1)}{3\beta^2(2\beta + 1)} x^2 + d x^{-2\beta - 1}$</td>
<td>$C \frac{2(\beta + 1)}{2\beta + 1} - d(\beta + 1)x^{-2\beta - 1}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\kappa = 0$ (Uniform Expansion)</th>
<th>$x = 0$ (Synchronous)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma$</td>
<td>$C \frac{2}{3\beta(2\beta + 1)} x^2 - dx^{-2\beta - 1}$</td>
<td>$C \frac{2(\beta + 1)(\beta - 2)}{3\beta^2(2\beta + 1)(\beta + 3)} x^2 - dx^{-2\beta - 1} + \frac{2(\beta + 1)}{\beta} x^{-\beta - 1}$</td>
</tr>
<tr>
<td>$\frac{x^2}{\beta} H_{X}$</td>
<td>$C \frac{1}{\beta(2\beta + 1)} x^2 - d \frac{3}{2} x^{-2\beta - 1}$</td>
<td>$C \frac{1}{\beta(2\beta + 1)} x^2 - d \frac{3}{2} x^{-2\beta - 1} + \frac{1}{\beta^2} x^{-\beta - 1}$</td>
</tr>
<tr>
<td>$\kappa/H$</td>
<td>$0$</td>
<td>$C \frac{2}{2(\beta - 1)} x^{-2\beta - 1}$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$-C \frac{2}{3\beta^2(\beta + 1)} x^2 - d \frac{1}{3(\beta \beta + 1)(2\beta - 1)} x^{-2\beta - 1}$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\varphi$</td>
<td>$C + d \frac{\beta}{2(\beta - 1)} x^{-2\beta - 1}$</td>
<td>$C + d \frac{\beta}{2(\beta - 1)} x^{-2\beta - 1} + g x^{-\beta - 1}$</td>
</tr>
<tr>
<td>$\delta$</td>
<td>$C \frac{2}{3\beta^2} x^2 + d \frac{1}{3(2\beta - 1)^2} x^{-2\beta - 3}$</td>
<td>$C \frac{2(\beta + 1)}{3\beta^2(\beta + 3)} x^2 + d \frac{\beta - 2}{3(2\beta - 1)(\beta - 4)} x^{-2\beta - 3} - g \frac{2(\beta + 1)}{\beta} x^{-\beta - 1}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\varphi = 0$ (Uniform Curvature)</th>
<th>$\epsilon = 0$ (Uniform Density)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma$</td>
<td>$C \frac{2(\beta + 1)}{\beta} - d \frac{\beta - 2}{2(\beta - 1)} x^{-2\beta - 1}$</td>
<td>$-C \frac{2(\beta + 1)}{3\beta^2(\beta + 1)} x^2 - dx^{-2\beta - 1}$</td>
</tr>
<tr>
<td>$\frac{x^2}{\beta} H_{X}$</td>
<td>$-C \frac{\beta + 1}{\beta^3(2\beta + 1)} x^2 - d \frac{3}{2} x^{-2\beta - 1}$</td>
<td>$C \frac{1}{\beta(2\beta + 1)} x^2 - d \frac{3}{2} x^{-2\beta - 1}$</td>
</tr>
<tr>
<td>$\kappa/H$</td>
<td>$-C \frac{2(\beta + 1)}{\beta} - d \frac{3(\beta + 1)}{2(2\beta - 1)} x^{-2\beta - 1}$</td>
<td>$C \frac{1}{\beta^2} x^2 + d \frac{1}{2\beta(2\beta - 1)} x^{-2\beta - 3}$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$-C \frac{\beta + 1}{\beta} + d \frac{\beta - 2}{2(2\beta - 1)} x^{-2\beta - 1}$</td>
<td>$C \frac{1}{3\beta^2} x^2 - d \frac{\beta - 2}{6\beta(\beta + 1)(2\beta - 1)} x^{-2\beta - 3}$</td>
</tr>
<tr>
<td>$\varphi$</td>
<td>$0$</td>
<td>$C + d \frac{\beta}{2(2\beta - 1)} x^{-2\beta - 1}$</td>
</tr>
<tr>
<td>$\delta$</td>
<td>$C \frac{2(\beta + 1)}{\beta} + d \frac{\beta + 1}{2(2\beta - 1)} x^{-2\beta - 1}$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

Note.—The leading behaviors valid on superhorizon scales are shown. This is a reproduction of Table 1 of H1, this time using $C$ which is conserved even under changing $\beta$ (considering only the growing mode) and using $(x/\beta)^2 H_{X}$. $C$ and $d$ are the constant coefficients of the growing and the decaying mode, respectively; $g$ appearing only in the SG, is a constant coefficient for the gauge mode. Since we used expansion in $x$, the solutions in the $w = 0$ case are not necessarily valid (see Table 5). Differences occur only in some of the decaying modes in the SG. Some divergences appearing at $\beta = -3$, $4(w = -\frac{1}{2}, -\frac{3}{2})$ are artifacts corrected in Table 3 and the case for $\beta = 1$ ($w = 1$) is separately considered in Table 4. Comparing with Table 3 and Table 4 one can see that the temporal behaviors in this table remain the same (except for some logarithmic dependences): $x \propto t^{1/\beta + 1} \propto a^{1/\beta}$.  

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TABLE 3

EXCLUDED CASES OF TABLE 2 FOR $\beta = -3, 4$

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\beta = -3$</th>
<th>$\beta = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma H$</td>
<td>$C x^2 (1 + \frac{1}{x} \ln x) - dx^2 - \frac{2}{3} dx$</td>
<td>$\frac{2}{7} C x^2 - dx - \frac{1}{7} (1 + \frac{1}{x} \ln x) + \frac{2}{3} x^2$</td>
</tr>
<tr>
<td>$\kappa H$</td>
<td>$C x^2 (\frac{3}{2} + \frac{1}{x} \ln x) + \frac{2}{7} dx + 2 x^2$</td>
<td>$\frac{2}{7} C x^2 + dx - \frac{1}{7} (1 + \frac{1}{x} \ln x) + \frac{2}{3} x^2$</td>
</tr>
<tr>
<td>$\delta$</td>
<td>$C x^2 (\frac{3}{2} + \frac{1}{x} \ln x) - \frac{2}{7} dx - \frac{1}{3} x^2$</td>
<td>$\frac{2}{7} C x^2 - dx - \frac{1}{7} (1 + \frac{1}{x} \ln x) - \frac{2}{3} x^2$</td>
</tr>
</tbody>
</table>

Note.—Divergences appearing in some of the SG solutions at $\beta = -3, 4$ in Table 2 are due to improper integration of $x^{-1}$ which are corrected in this table. We present only the case in which differences occur from Table 2, $\gamma$ is Euler’s constant.

TABLE 4

LARGE-SCALE SOLUTION FOR $w = 1$ ($\beta = \frac{1}{2}$)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\Psi = 0$ (Comoving)</th>
<th>$\chi = 0$ (Zero Shear)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma H$</td>
<td>$C x^2 - \frac{1}{3}dx$</td>
<td>$\frac{3}{4} C - \frac{1}{2} dx$</td>
</tr>
<tr>
<td>$\left(\frac{x}{H}\right)^2 H_X$</td>
<td>$C x^2 - \frac{1}{3} dx$</td>
<td>$-\frac{2}{3} C + \frac{1}{3} dx$</td>
</tr>
<tr>
<td>$\kappa H$</td>
<td>$C x^2 - \frac{1}{3} dx$</td>
<td>$-\frac{2}{3} C - \frac{1}{3} dx$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$\frac{1}{4} d \left(\gamma + \ln \frac{x}{2}\right)$</td>
<td>$\frac{1}{4} C + \frac{1}{4} dx$</td>
</tr>
<tr>
<td>$\delta$</td>
<td>$2 C x^2 + d$</td>
<td>$\frac{3}{4} C - \frac{1}{2} dx$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\kappa = 0$ (Uniform Expansion)</th>
<th>$\alpha = 0$ (Synchronous)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma H$</td>
<td>$\frac{3}{4} C x^2 - d$</td>
<td>$-\frac{1}{3} C x^2 - d - 6 g x^{-3/2}$</td>
</tr>
<tr>
<td>$\left(\frac{x}{H}\right)^2 H_X$</td>
<td>$C x^2 - \frac{1}{3} dx$</td>
<td>$C x^2 - \frac{1}{3} dx + 4 g x^{1/2}$</td>
</tr>
<tr>
<td>$\kappa H$</td>
<td>$0$</td>
<td>$\frac{16}{7} C x^2 - \frac{4}{7} dx - \frac{20}{21} \left(\gamma + \ln \frac{x}{2}\right) + 9 g x^{-3/2}$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$-\frac{16}{9} C x^2 + \frac{4}{9} dx \left(-\frac{2}{3} + \gamma + \ln \frac{x}{2}\right)$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\phi$</td>
<td>$C - \frac{1}{4} d \left(-\frac{2}{3} + \gamma + \ln \frac{x}{2}\right)$</td>
<td>$C - \frac{1}{4} d \left(-\frac{2}{3} + \gamma + \ln \frac{x}{2}\right) + g x^{-3/2}$</td>
</tr>
<tr>
<td>$\delta$</td>
<td>$\frac{8}{3} C x^2 - \frac{2}{3} dx \left(-\frac{2}{3} + \gamma + \ln \frac{x}{2}\right)$</td>
<td>$\frac{8}{7} C x^2 - \frac{2}{7} dx \left(-\frac{55}{224} + \gamma + \ln \frac{x}{2}\right) - 6 g x^{-3/2}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\phi = 0$ (Uniform Curvature)</th>
<th>$\epsilon = 0$ (Uniform Density)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma H$</td>
<td>$6 C - \frac{3}{2} d \left(\gamma + \ln \frac{x}{2}\right)$</td>
<td>$-2 C x^2 - d$</td>
</tr>
<tr>
<td>$\left(\frac{x}{H}\right)^2 H_X$</td>
<td>$-3 C x^2 - \frac{1}{3} dx$</td>
<td>$C x^2 - \frac{1}{3} dx$</td>
</tr>
<tr>
<td>$\kappa H$</td>
<td>$C - \frac{3}{4} d \left(-\frac{2}{3} + \gamma + \ln \frac{x}{2}\right)$</td>
<td>$4 C x^2 - dx \left(-\frac{2}{3} + \gamma + \ln \frac{x}{2}\right)$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$-\frac{3}{4} C + \frac{3}{4} d \left(\gamma + \ln \frac{x}{2}\right)$</td>
<td>$\frac{4}{3} C x^2 - \frac{1}{3} dx \left(\gamma + \ln \frac{x}{2}\right)$</td>
</tr>
<tr>
<td>$\phi$</td>
<td>$0$</td>
<td>$C - \frac{1}{4} d \left(-\frac{2}{3} + \gamma + \ln \frac{x}{2}\right)$</td>
</tr>
<tr>
<td>$\delta$</td>
<td>$6 C + \frac{3}{2} d \left(-\frac{2}{3} + \gamma + \ln \frac{x}{2}\right)$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

Note.—Solutions for $\beta = \frac{1}{2}$ ($w = 1$) case valid on superhorizon scales are shown. For Bessel functions with integer index we need a different form of the asymptotic expansion from the noninteger case. One can check that to the leading large-scale order the only difference occurs for $\beta = \frac{1}{2}$ which lead to Bessel equation with index 1. Notice that the temporal behavior of all modes are coincident with Table 2 (except for some logarithmic dependences).
3.3.2. Conservation Variables

From Table 2, considering only the growing mode, we can read
\[ \phi_\varphi = \phi_\chi = \phi_\xi = C, \]
which follow because on superhorizon scales \( \varphi \) is dominant over the other corresponding variable in each gauge (compare with Table 5 which will be valid on any scale in a pressureless medium). By definition \( \zeta = \phi_\varphi \), \( \Phi_\varphi = \phi_\varphi - (aH/\dot{a})\varphi \), (sometimes we neglect the spatial harmonic function \( Y \)) is often used in the literature as possessing a similar conservation property; Bardeen (1980) used \( h_\varphi \) which was denoted \( h_\chi \) in Bardeen et al. (1983). Equation (41) can be expressed as follows: in the CG, the UEG, the SG, and the UDG, on superhorizon scales we have \( \varphi = C \) which is conserved independent of the changes in the background equation of state.\(^1\) For all these gauges, the dominant decaying mode of \( \varphi \) has cancelled. Even considering the decaying mode (neglecting the gauge-mode in the SG) we have
\[ \phi_\varphi = \phi_\chi = \phi_\xi = C. \]  
(42)

Even in \( \phi_\varphi \), the dominant decaying mode cancels.

This is not the case in the ZSG where we have same order of variables (see eq. [44]). We may also note that in both the UCG (\( H_\varphi \) for the shear variable) and the ZSG the growing modes of every variable have the same order and are in fact constant in time for a given constant \( w \) stage. In the UCG case we have (see also Table 8)
\[ \frac{1}{2} \frac{\delta}{\dot{\delta}} = -\frac{\alpha}{\dot{\alpha}} = -\frac{1}{3} \frac{k}{\dot{k}} = -\frac{2\beta + 1}{\dot{\beta}} \frac{H_\varphi}{H} = 4\pi G \frac{\Psi}{H} = \frac{C}{\dot{\beta}} + \frac{1}{\dot{\beta}} = \frac{C}{2} (w + 1). \]
(43)

In summary, with the exception of the ZSG (and the UCG) the growing mode of \( \varphi \) remain constant independent of changes in the background equation of state, and the dominant decaying mode vanishes. Similar results more generally valid for changing \( p(\rho) \) on superhorizon scales will be presented from another viewpoint in § 3.7.

3.3.3. Comments on Zero-Shear Gauge Case

In the ZSG, the growing modes of each variable remains constant for each constant \( w \) domain. From Table 2, the growing modes behave as
\[ \frac{1}{2} \frac{\delta}{\dot{\delta}} = -\alpha = -\frac{1}{3} \frac{k}{\dot{k}} = 4\pi \frac{\Psi}{H} = \frac{C}{\dot{\beta}} + \frac{1}{\dot{\beta}} \]
\[ = \frac{C}{2} \left[ 1 + \frac{2}{3(1 + w)} \right]^{-1}, \]
(44)

which are all time-independent, but \( \beta \)-dependent. (Due to a signature difference, the \( \Phi \) and \( \Psi \) used in Linde 1990 and Mukhanov et al. 1992 are related to ours as \( \Phi = \Phi_\varphi \) and \( \Psi = -\Phi_\varphi = -\varphi_\varphi \)) where we neglect the harmonic function \( Y \); see eq. [17]). By neglecting anisotropic pressure, from equation (11) we have \( \alpha = -\varphi \). The method using \( \varphi_\varphi \) in deriving the inflationary spectrum is closely related to the method using the conservation property of \( \zeta \) or \( \rho_\varphi \)'s in other gauges which simply become \( C \). As will be shown in equations (63) and (68), \( \rho_\varphi \)'s in other gauges in equation (41) remain constant even in a transition of the background equation of state, whereas \( \varphi_\varphi \) changes to balance the constancy of the GI \( \rho_\varphi \)'s in equation (41) according to equations (63) and (68). Still, \( \varphi_\varphi \) shows consistent behavior with the Newtonian potential perturbation in any scale. Again, what is conserved is the integration constant \( C \) which on superhorizon scales can be identified with \( \varphi \) in many gauge choices (eq. [41]) where \( \zeta \) is one of them. We also note that the decaying modes of every variable in this gauge is of \( x^{-3/2} \) order which is at least \( x^{-2}(\gg 1) \) larger than the behavior of the same variables in every other gauge.

3.4. Exact Solution in \( p = 0 \) Era

Since equation (36) is an expansion in \( c_s, \chi \), which vanishes in the \( w = 0 (\beta = 2) \) case, on superhorizon scales the results presented in Table 2 are not necessarily applicable for MDE. Instead, using the exact solution in equation (29) we can derive solutions in an exact form valid on any scale; more precisely, outside Jeans scale which is negligible compared to the horizon scale in MDE. The solutions are presented in Table 5. On superhorizon scales, except for some decaying mode in the SG case, all the rest of the dominating modes behave the same way as may be deduced from Table 2. As noted before, the difference in the SG occurs because for some variables the cancellation of the dominant decaying mode which occurred for general \( w \) (thus using eq. [36]) does not occur in \( w = 0 \) case (use eq. [29]).

The solutions in the CG coincide exactly with the one in the SG, except for the fact that no gauge mode appears in the CG. This is due to the fact that in a pressureless medium one can take the CG and the SG simultaneously; imposing the CG conditions on equation (14) gives \( \alpha = 0 \); imposing the SG condition on equation (14) gives an equation for \( \Psi \) which describes only the gauge mode (see Table 5). In both the SG and CG cases, from equations (12)–(14) we can derive
\[ \frac{1}{2} \frac{\delta}{\dot{\delta}} + 2H\frac{\delta}{\dot{\delta}} = 4\pi G \mu \delta = 0. \]
(45)

Since, in the SG case, the time evolution of the gauge mode of \( \delta \) behaves the same way as the decaying mode, equation (45) ends up in second-order differential equation. Equation (45) for \( \delta \) remains the same in the Newtonian case for \( \delta \rho/\rho (\rho \) is a mass density).

From Table 5 one can read
\[ \zeta = C(1 + \frac{1}{3}x^2) + \frac{1}{2} dx^{-3}. \]  
(46)

Since \( x = 2k/(aH) \), on superhorizon scales \( (x \ll 1) \) and on subhorizon scales \( (x \gg 1) \) the growing modes of \( \zeta \) become \( \zeta = C \) and \( \zeta = (1/30)C x^2 \), respectively. Thus, except for obviously exceptional gauges, \( \zeta \) is dominated on superhorizon scales by \( \varphi \) which remains constant, and on subhorizon scales by \( \delta/3 \) which grows as \( \propto x^2 \propto t^{3/2} \). On subhorizon scales, in each gauge, the decaying mode is also dominated by \( \delta/3 \propto x^{-3} \propto t^{-1} \).
Except for the UDG, the growing mode of $\phi$ is conserved on any scale (the dominant decaying mode has cancelled). This is true even during the transition between superhorizon and subhorizon where only the UEG is the exception. In the UEG, $\Psi$ on superhorizon scales behaves as $\Psi|_{\text{CG}}$, whereas on subhorizon scales it behaves as $\Psi|_{\text{ZSG}}$; $\Psi$ on subhorizon scales also follows $\Psi|_{\text{ZSG}}$, whereas $\delta$ on subhorizon scales follows $\delta|_{\text{CG}}$, etc. In the SG, the gauge mode of $\delta$ has the same time dependence as the decaying mode; this is not necessarily the case with the other variables.

3.4.1. Comments on Newtonian Limit

We can notice that on subhorizon scales ($x \gg 1$) the density perturbations in many gauge choices, with the exception of the UDG, behave as

$$
\delta = cx^2 + dx^{-3} \propto ct^{2/3}, dt^{-1},
$$

which is in accord with the well-known result in Newtonian theory; $c = C/10$. On superhorizon scales, due to the metric dominance in some gauges, the behavior of $\delta$ differs depending on the specific gauge choice. In the UDG, since $\delta_{\Psi} = -3H\Psi$, one way of understanding the situation is to consider that in this gauge the information about $\epsilon$ is transferred into $-3H\Psi$. Thus we have $\delta_{\Psi} = -8\pi G\Psi/H$; remember that physically measurable quantities should be $\Gamma$.

In MDE $\delta_{\Psi}, v_x$, and $\phi_x$ become

$$
\delta_{\Psi} = cx^2 + dx^{-3}, \quad v_x Y = -2cx + 3dx^{-4},
$$

$$
\phi_x = 6c + 6dx^{-5}. \quad (48)
$$

We can compare these with the Newtonian results (Peebles 1980; Hwang 1992). One may introduce perturbations in the mass density and velocity as $\rho \rightarrow \tilde{\rho} + \delta \rho Y, v^a \rightarrow \tilde{v}^a + \delta v^a$. In comoving coordinates, the mass conservation equation implies $\delta \rho = -(a/k)(\delta \rho/p)$. Thus, if we identify $(\delta \rho/p)Y = cx^2 + dx^{-3}$, since $\delta \tilde{v} = -d(\delta \rho/p)/dx$, we have $\delta \tilde{v}Y = -2cx + 3dx^{-4}$. In addition, from the Poisson's equation, $-\delta \Phi = 4\pi G(a/k)^2\delta \rho$, we have $-4\delta \Phi = 6x^{-2}\delta \rho/p = 6c + 6dx^{-5}$. These can be compared with equation (48). In fact, in the Newtonian case it was shown in Hwang (1992) that

$$
h^2 \rho_{\delta} = \delta \rho Y_x, \quad \sigma = \frac{1}{\sqrt{2}} \frac{\sigma_{Y\delta}}{a \delta t} = \frac{k}{\sqrt{2}} \frac{1}{Y_x Y_{x\delta}}. \quad (49)
$$
where derivatives and the harmonics are based on comoving coordinates \( h_{\delta} \) is a three-space metric which becomes \( \delta_{\delta} \) in Cartesian coordinates); whereas in general relativity we have equations (19) and (20). These suggest that “the GI variables \( \phi, v, \) and \( \phi_s \) correspond to the perturbed density, velocity, and potential variables \( \delta \phi / \rho, \delta v, \) and \( -\delta \Phi, \) respectively in Newtonian case” (we neglect \( Y_s \)).

If we take the ZSG \( (\chi = 0) \), which is the same as using \( \phi_s \), etc., from equation (11), we have \( \chi = - \phi - \sigma \). Thus, if we also neglect the anisotropic pressure \( (\sigma = 0) \) we have \( \chi = - \phi \). Since \( \chi = 0 \) condition imposes a relation between \( \beta \) and \( \gamma \) (see eq. [2]), and since \( \beta \) and \( \gamma \) are separately spatially gauge-dependent variables, by choosing an appropriate spatial gauge condition can remove their effect; remember that our set of equations does not involve the variable dependent of the spatial gauge transformation. Thus, in this case the metric in equation (1) becomes

\[
d s^2 = -a^2(1 - 2\phi)dt^2 + a^2(1 + 2\phi) dx^2 dx^4,
\]

which is in the metric form of the Newtonian limit (§ 6 of Peebles 1980). The variable \( \phi \) (since we are assuming the ZSG \( \phi_s \) is equivalent) can be identified with \( -\Phi \) where \( \Phi \) is the perturbed gravitational potential in Newtonian theory (for the correspondence considering \( K \), see eq. [21]). This reinforces connection of the GI variables in equation (48) to corresponding Newtonian ones (see p. 46 of Kodama & Sasaki 1984). For complementary discussions, see HH11.

3.5. Exact Solution for \( p = -\mu/3 \), Considering General \( K \)

As noted below equation (27), the case with \( p = \mu/3 \) should be treated separately. Since in this case \( \mu \) evolves as \( a^{-2} \) which coincides with the behavior of the curvature term, we can solve the equation including general \( K \) in it. This includes the case with no active gravitational mass (i.e., \( \mu + 3p = 0 \)) and also includes the stage when the curvature term dominates in the open \( (K = -1) \) FLRW model; a string-dominated universe is also known to have a period with \( \chi \propto t \) (Vilenkin 1984; Turner 1985). Defining \( \mu = Ma^{-2} \) where \( M \) is a constant, equations (6) can be solved with the result

\[
a = \left( \frac{M}{2} - K \right)^{1/2} t.
\]

In this case the CG equation (25) becomes

\[
\bar{\delta} + 3H\bar{\delta} - \frac{k^2}{3a^2} \bar{\delta} = 0.
\]

The solutions can be written as

\[
\delta = \bar{\epsilon} t = c_1 t^{x_1} + c_2 t^{x_2},
\]

where

\[
z_1, z_2 = -1 \pm \left( 1 + \frac{k^2}{M - 3K} \right)^{1/2}.
\]

Following the same method described in § 3.1 we can derive the behavior of the solutions in all different gauges. The general solutions valid in all scales are presented in Table 6.

3.6. \( p \) Exactly Equal to \( -\mu \)

Due to the cancellations of the \( \mu + p \) terms in equations (13) and (14), the case when \( p = -\mu \) also requires special treatment.

In this case equations (13) and (14) can be combined to give (not using any gauge condition)

\[
\ddot{\epsilon} + 5H\dot{\epsilon} - \frac{k^2}{a^2} \epsilon = 0,
\]

where \( a \propto e^{Ht} \) and \( H = \text{constant} \) follow from equation (6). This is not valid in the CG and the UDG because we have \( \Psi = \epsilon = 0 \) and the other variables cannot be determined. From equations (8)–(12) we can show

\[
\kappa = \frac{k^2}{a^2} \chi, \quad \varphi = H\chi, \quad \chi = \dot{x},
\]

where \( \chi \) can be arbitrary; thus, in this exact \( p = -\mu \) case these two gauge choices do not behave properly. However, in all other gauge choices introducing \( f \) similarly as in equation (26) with \( \beta = -1 \), i.e., \( f \equiv x^{-3} \delta \), we have

\[
x^2f'' + 2xf' - (x^2 + 2 \times 3) f = 0,
\]

which is a modified spherical Bessel equation of order 2. The solution can be written as

\[
f = \tilde{a}j_2(\tilde{x}) + \tilde{b}n_2(\tilde{x}),
\]

where

\[
j_2(\tilde{x}) = -\left( \frac{3}{x^2} + \frac{1}{x^3} \right) \sinh x + \frac{3}{x^2} \cosh x,
\]

\[
n_2(\tilde{x}) = \frac{1}{i} \left[ \left( \frac{3}{x^2} + \frac{1}{x^3} \right) \cosh x - \frac{3}{x^2} \sinh x \right].
\]

In the large-scale \( (x \ll 1) \) we have

\[
f = \tilde{e} \left( x^2 + \frac{x^4}{14} + \cdots \right) + \tilde{d} \left( x^{-3} - \frac{x^{-1}}{6} + \frac{x}{24} + \cdots \right),
\]

where, in accordance with our general definition in equation (37) with \( \beta = 2 \), we introduced

\[
\tilde{e} \equiv -\frac{1}{15} \tilde{a}, \quad \tilde{d} = \frac{3}{i} \tilde{b}.
\]

Except for the CG and the UDG, \( \epsilon \) and \( \Psi \) can be solved independent of the gauge choices as

\[
\delta = x^2f, \quad 8\pi G \frac{\Psi}{H} = -3x^2f'' - 9xf.
\]

In the large scale we have

\[
\delta = \ddot{\epsilon} x^5 + \ddot{d}, \quad 8\pi G \frac{\Psi}{H} = -15\ddot{\epsilon} x^3 + \ddot{d}.
\]

The rest of the complete solutions are presented in Table 7.

In contrast to the \( p = 0 \) case, the vanishing of some terms due to the condition \( p + \mu = 0 \) are not realistic. For example the \( \alpha \) terms in equations (13) and (14) both occur as \( (\mu + p) \alpha \). (The \( w = -1 \) case considered in Hwang 1991a [H3 hereafter] should be considered as the case with \( w \) very close to \(-1 \) which is the usually considered case in inflationary scenarios.) Thus we emphasize that the solutions presented in this section are valid when \( p \) is strictly equal to \(-\mu \) which is in a sense a degenerate case.
TABLE 6

EXACT SOLUTION FOR \( w = -\frac{1}{3} \) (\( \beta \) Diverges) CONSIDERING GENERAL \( K \)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>( \Psi = 0 ) (Comoving)</th>
<th>( \chi = 0 ) (Zero Shear)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \gamma \Psi / H )</td>
<td>0</td>
<td>( \frac{M}{k^2 - 3K} ) ( (z + 1) \hat{t} \hat{r} )</td>
</tr>
<tr>
<td>( H_\gamma )</td>
<td>( \frac{M - 3K}{k^2 - 3K} \frac{1}{2} (1 + z) \hat{r} \hat{t} )</td>
<td>0</td>
</tr>
<tr>
<td>( \kappa / H )</td>
<td>( \frac{\lambda}{2} (1 + z) \hat{r} \hat{t} )</td>
<td>( -\frac{M}{k^2 - 3K} \frac{3}{2} (z + 1) \hat{r} \hat{t} )</td>
</tr>
<tr>
<td>( a )</td>
<td>( \frac{\lambda}{2} \hat{t} \hat{r} )</td>
<td>( -\frac{M}{k^2 - 3K} \frac{1}{2} \hat{r} \hat{t} )</td>
</tr>
<tr>
<td>( \phi )</td>
<td>( \left[ \frac{M - 3K}{k^2 - 3K} (1 + z) + \frac{M}{k^2 - 3K} \right] \frac{1}{2} \hat{t} \hat{r} )</td>
<td>( \frac{M}{k^2 - 3K} \frac{1}{2} \hat{t} \hat{r} )</td>
</tr>
<tr>
<td>( \delta )</td>
<td>( \hat{r} \hat{t} )</td>
<td>( \left[ \frac{M - 3K}{k^2 - 3K} (z + 1) + \frac{1}{2} \right] \hat{t} \hat{r} )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameter</th>
<th>( \kappa = 0 ) (Uniform Expansion)</th>
<th>( \chi = 0 ) (Synchronous)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \gamma \Psi / H )</td>
<td>( \frac{M (M - 3K)(z + 2) + k^2}{k^2} \frac{1}{M + k^2 - 3K} \frac{1}{z} \hat{r} \hat{t} )</td>
<td>( \frac{M}{k^2 - 3K} \left[ \frac{z(z + 2) - 3K}{M - 3K} \right] \frac{1}{1 + z} \hat{t} \hat{r} - 2 \frac{M}{M - 3K} t^{-1} \hat{g} )</td>
</tr>
<tr>
<td>( H_\gamma )</td>
<td>( \frac{M - 3K}{k^2 - 3K} \frac{M}{M + k^2 - 3K} \frac{k^2}{4} \hat{r} \hat{t} )</td>
<td>( \frac{M}{k^2 - 3K} \frac{1}{2} \hat{t} \hat{r} )</td>
</tr>
<tr>
<td>( \kappa / H )</td>
<td>0</td>
<td>( \frac{M}{k^2 - 3K} \left[ \frac{k^2}{M - 3K} - z(z + 2) \right] \frac{3}{2} \frac{1}{1 + z} \hat{t} \hat{r} + \frac{M + k^2 - 3K}{M - 3K} 3t^{-1} \hat{g} )</td>
</tr>
<tr>
<td>( a )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \phi )</td>
<td>( \frac{(M - 3K)(z + 2) + k^2}{M + k^2 - 3K} \frac{1}{k^2 - 3K} \frac{2}{k^2 - 3K} \frac{1}{z + 2} \hat{r} \hat{t} )</td>
<td>( \frac{M}{k^2 - 3K} \hat{t} \hat{r} + t^{-1} \hat{g} )</td>
</tr>
<tr>
<td>( \delta )</td>
<td>( \frac{(M - 3K)(z + 2) + k^2}{M + k^2 - 3K} \hat{r} \hat{t} )</td>
<td>( \left[ \frac{M - 3K}{k^2 - 3K} (z + 2) + z + 2 - \frac{k^2}{k^2 - 3K} \right] \frac{1}{1 + z} \hat{t} \hat{r} - 2t^{-1} \hat{g} )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameter</th>
<th>( \phi = 0 ) (Uniform Curvature)</th>
<th>( \epsilon = 0 ) (Uniform Density)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \gamma \Psi / H )</td>
<td>( \frac{M}{k^2 - 3K} \left[ z + 2 + \frac{3K}{M - 3K} \right] \hat{r} \hat{t} )</td>
<td>( -\frac{M}{M - 3K} \hat{t} \hat{r} )</td>
</tr>
<tr>
<td>( H_\gamma )</td>
<td>( -\frac{M}{k^2 - 3K} \frac{1}{2} \hat{t} \hat{r} )</td>
<td>( -\frac{M - 3K}{k^2 - 3K} \left[ z + 2 - \frac{M - k^2}{M - 3K} \right] \frac{1}{2} \hat{t} \hat{r} )</td>
</tr>
<tr>
<td>( \kappa / H )</td>
<td>( -\frac{M}{k^2 - 3K} \left( z + 2 + \frac{k^2}{M - 3K} \right) \frac{3}{2} \hat{t} \hat{r} )</td>
<td>( \left[ z + 2 + \frac{k^2}{M - 3K} \right] \frac{3}{2} \hat{t} \hat{r} )</td>
</tr>
<tr>
<td>( a )</td>
<td>( -\frac{M}{k^2 - 3K} \frac{1}{2} (z + 2) \hat{r} \hat{t} )</td>
<td>( \frac{1}{2} (z + 2) \hat{r} \hat{t} )</td>
</tr>
<tr>
<td>( \phi )</td>
<td>0</td>
<td>( \frac{M - 3K}{k^2 - 3K} \left[ z + 2 + \frac{k^2}{M - 3K} \right] \frac{1}{2} \hat{t} \hat{r} )</td>
</tr>
<tr>
<td>( \delta )</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Note.—Exact solution for \( w = -\frac{1}{3} \) (\( \beta \) diverges) considering general \( K \). Solutions written as \( \gamma(z) \hat{t} \hat{r} \) actually represent \( \gamma(z_1) \hat{t} \hat{r} + \gamma(z_2) \hat{t} \hat{r} \) where \( z_1, z_2 = -1 \pm \sqrt{1 + k^2(M - 3K)^{1/2}} \) (where \( \mu = Ma^{-2} \)). One can notice that every nonvanishing and nongauge mode behaves as \( t^\epsilon \); we use \( H_\gamma \).
TABLE 7

The Exact and the Large-Scale Asymptotic Solution for \( w = -1 \)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>( \chi = 0 )</th>
<th>( \kappa = 0 )</th>
<th>( \alpha = 0 )</th>
<th>( \varphi = 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \left( \frac{\kappa}{H} \right)^{\chi} )</td>
<td>0</td>
<td>( -\frac{3}{5}x^3f' - \frac{1}{2}x^2f )</td>
<td>( 3x^2 \int f dx - \frac{5}{2}x^2f' - \frac{5}{2}x^2f )</td>
<td>( -\frac{3}{2}x^3f' - \frac{3}{2}x(9 + x^2)f )</td>
</tr>
<tr>
<td>( \kappa/H )</td>
<td>( \frac{3}{5}x^3f' + \frac{1}{2}x^2f )</td>
<td>0</td>
<td>( 3x^2 \int f dx )</td>
<td>( -\frac{3}{2}x^3f' )</td>
</tr>
<tr>
<td>( x )</td>
<td>( -\frac{3}{5}x^3f' - \frac{3}{5}x^3(9 + x^2)f )</td>
<td>3xf</td>
<td>0</td>
<td>( \frac{3}{2}x^3f' + \frac{1}{2}x^2f )</td>
</tr>
<tr>
<td>( \varphi )</td>
<td>( \frac{3}{5}x^3f' + \frac{1}{2}x^3(9 + x^2)f )</td>
<td>( \frac{1}{2}x^3f' )</td>
<td>( 3 \int f dx + \frac{2}{5}x^3f )</td>
<td>0</td>
</tr>
</tbody>
</table>

The large scale asymptotic forms

<table>
<thead>
<tr>
<th>Parameter</th>
<th>( \chi = 0 )</th>
<th>( \kappa = 0 )</th>
<th>( \alpha = 0 )</th>
<th>( \varphi = 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \left( \frac{\kappa}{H} \right)^{\chi} )</td>
<td>0</td>
<td>( -\frac{3}{5}x^3 \frac{d}{dx} + \frac{1}{2} \frac{d}{dx} )</td>
<td>( -\frac{3}{5}x^3 - \frac{3}{2}x(1 + \frac{3}{2} \ln x)d + \frac{5}{2}x^3 )</td>
<td>( -\frac{3}{2}x^3 - \frac{3}{4} dx^2 )</td>
</tr>
<tr>
<td>( \kappa/H )</td>
<td>( \frac{3}{5}x^3 \frac{d}{dx} - 2d )</td>
<td>0</td>
<td>( \frac{3}{2}x^3 - \frac{3}{4} dx^2 )</td>
<td>0</td>
</tr>
<tr>
<td>( x )</td>
<td>( -\frac{3}{5}x^3 \frac{d}{dx} - \frac{1}{2} dx^2 )</td>
<td>( 3x^3 + 3 dx^2 - 2 d )</td>
<td>( \frac{3}{2}x^3 \frac{d}{dx} - \frac{3}{4} )</td>
<td>0</td>
</tr>
<tr>
<td>( \varphi )</td>
<td>( \frac{3}{5}x^3 \frac{d}{dx} + \frac{1}{2} dx^2 )</td>
<td>( \frac{3}{2}x^3 - \frac{3}{4} (1 + 2 \ln x)d + \frac{3}{4} x^3 )</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Note: — The exact and the large-scale asymptotic solution for \( w = -1 \) is a degenerate case. We omit the CG and the UDG cases which are inappropriate in this case where \( \rho \) is exactly equal to \( -\mu \). For the other gauges \( \epsilon \) and \( \psi \) can be solved independent of the gauge choices (eqs. [61], [62]); we present the rest in this table: \( f = \tilde{a} \tilde{g}(\chi) + b_n \tilde{g}(\chi) \).

3.7. Large-Scale Integral Form Solutions

In this section we consider general \( \rho(\mu) \), and we also retrieve general \( K \) and \( A \). It is well known that for an ideal-fluid medium, in the large scale, the evolution of a GI potential variable \( \varphi_z \) (which is \( \varphi \) in the ZSG) can be represented in terms of the integral-form solution (HV1) as:

\[
\varphi_z = -\frac{4}{3} \frac{d}{dx} \left[ \left( 1 - \frac{H}{a} \right) \int \left( 1 - \frac{a}{K} \right) dt \right],
\]

where \( C \) is a constant coefficient: an integration constant. The coefficient \( C \) represents the amplitude of the growing mode, and is the same variable as in equation (38); the decaying mode is absorbed into the lower bound of the integration. For a simple derivation in our notation see § V of HV1 and § 3.4 of H2 where some reservation on \( K \) term in general \( \rho \) case was noted. Knowing \( \varphi_z \) using equation (21) one can derive \( \varphi_x \), and from these two variables one can determine the evolution of all the rest of the variables in every gauge as we have done in previous sections.

The importance of this solution is apparent. If we know only the evolution of the background scale factor \( a(t) \), by simple differentiation and integration one can easily determine the evolution of the perturbed potential variables. From equation (21) we can derive \( \delta \rho \) which allows us to go through the same method described in § 3.1, and thus generate all the other general integral-form solutions in any gauge. We present the complete solutions in Table 8, assuming \( K = 0 = A \). Although many of the solutions presented in Table 8 are also applicable on general scale in pressureless medium, in general they are applicable on superhorizon scales.

In the SG case, since \( \varphi_z = C(a^{-1} \int a dt) \), we get

\[
\chi = \int \varphi_z a dt = C \frac{1}{a} \int a dt + \tilde{g},
\]

where the gauge mode (denoted by \( \tilde{g} \)) comes from the lower bound of integration of \( \varphi_z dt \). Since the dominating-mode function vanishes for some variables, in those cases we present the order of the nonvanishing dominating order term.

Using the solutions in Table 8 one can check the comments made in § 3.3. One can see that some GI variables behave as:

\[
\delta \varphi = C \frac{2}{3} \left( 1 - \frac{H}{a} \int a dt \right) \left( \frac{k}{aH} \right)^2,
\]

\[
v_z Y = -C \frac{H}{a} \int a dt \left( \frac{k}{aH} \right), \quad \zeta = C.
\]

The lower bound of integrations in Table 8 indicates the evolution of the decaying mode. The solution without the integration indicates that the dominant decaying mode has cancelled out.

As long as the ideal-fluid assumption holds, equation (63) includes the general case where the background equation of state changes as \( \rho(\mu) \). For general \( \omega \), equation (63) is valid near a flat background (H2). Equation (63) is valid even considering \( A \) in general, and considering \( K \) for a pressureless background medium. In a pressureless medium equation (63) is valid outside Jeans scale which is negligible compared to the horizon scale; in this case we have a simpler form to be derived in

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### Table 8
Solutions valid on superhorizon scales in the integration

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\Psi = 0$ (Comoving)</th>
<th>$\chi = 0$ (Zero Shear)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma \Psi / H$</td>
<td>$0$</td>
<td>$-2C \frac{H}{a} \int \frac{r}{a} dt$</td>
</tr>
<tr>
<td>$H \chi$</td>
<td>$C \frac{H}{a} \int \frac{r}{a} dt$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\kappa / H$</td>
<td>$C \frac{H}{a} \int \frac{r}{a} \left( \frac{k}{aH} \right)^2 dt$</td>
<td>$3C \frac{H}{a} \int \frac{r}{a} dt$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$-C \frac{2}{3} \frac{r^2}{1 + w} \left( 1 - \frac{H}{a} \int \frac{r}{a} dt \right) \left( \frac{k}{aH} \right)^2$</td>
<td>$-C \left( 1 - \frac{H}{a} \int \frac{r}{a} dt \right)$</td>
</tr>
<tr>
<td>$\phi$</td>
<td>$C$</td>
<td>$C \left( 1 - \frac{H}{a} \int \frac{r}{a} dt \right)$</td>
</tr>
<tr>
<td>$\delta$</td>
<td>$C \frac{2}{3} \left( 1 - \frac{H}{a} \int \frac{r}{a} dt \right) \left( \frac{k}{aH} \right)^2$</td>
<td>$-2C \frac{H}{a} \int \frac{r}{a} dt$</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\kappa = 0$ (Uniform Expansion)</th>
<th>$\alpha = 0$ (Synchronous)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma \Psi / H$</td>
<td>$2 \frac{C}{3} \frac{H}{a} \int \frac{r}{a} \left( \frac{k}{aH} \right)^2 dt$</td>
<td>$\frac{2}{3} \left[ \frac{1}{a} \int \left( \frac{k}{aH} \right)^2 \phi \right] dt = \frac{2}{3} \left[ \frac{1}{a} \int \left( \frac{k}{aH} \right)^2 \phi \right] dt + 2 \frac{H}{H} \dot{\gamma}$</td>
</tr>
<tr>
<td>$H \chi$</td>
<td>$C \frac{H}{a} \int \frac{r}{a} dt$</td>
<td>$H \int \frac{\dot{\phi}}{a} dt = C \frac{H}{a} \int \frac{r}{a} dt + H \dot{\gamma}$</td>
</tr>
<tr>
<td>$\kappa / H$</td>
<td>$0$</td>
<td>$-3 \left[ \frac{1}{a} \int \left( \frac{k}{aH} \right)^2 \phi \right] dt = \frac{3}{3} \left[ \frac{1}{a} \int \left( \frac{k}{aH} \right)^2 \phi \right] dt - 3 \frac{H}{H} \dot{\gamma}$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$-C \frac{2}{3} \frac{1 + 3r^2}{1 + w} \left( \frac{k}{aH} \right)^2$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\phi$</td>
<td>$C \frac{1}{a} \left( \frac{r}{a} \right)^2 \phi dt = C + H \dot{\gamma}$</td>
<td></td>
</tr>
<tr>
<td>$\delta$</td>
<td>$C \frac{2}{3} \left( \frac{k}{aH} \right)^2$</td>
<td>$2 \left[ \frac{1}{a} \int \left( \frac{k}{aH} \right)^2 \phi \right] dt = \frac{2}{3} \left[ \frac{1}{a} \int \left( \frac{k}{aH} \right)^2 \phi \right] dt + 2 \frac{H}{H} \dot{\gamma}$</td>
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<th>$\epsilon = 0$ (Uniform Density)</th>
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</thead>
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<tr>
<td>$\gamma \Psi / H$</td>
<td>$3C(1 + w)$</td>
<td>$-C \frac{2}{3} \left( 1 - \frac{H}{a} \int \frac{r}{a} dt \right) \left( \frac{k}{aH} \right)^2$</td>
</tr>
<tr>
<td>$H \chi$</td>
<td>$-C \left( 1 - \frac{H}{a} \int \frac{r}{a} dt \right)$</td>
<td>$C \frac{H}{a} \int \frac{r}{a} dt$</td>
</tr>
<tr>
<td>$\kappa / H$</td>
<td>$-C \frac{2}{3} \left( 1 + w \right)$</td>
<td>$C \left( \frac{k}{aH} \right)^2$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$-C \frac{1}{3} \left( \frac{k}{aH} \right)^2$</td>
<td>$C \frac{1}{3} \left( \frac{k}{aH} \right)^2$</td>
</tr>
<tr>
<td>$\phi$</td>
<td>$0$</td>
<td>$C$</td>
</tr>
<tr>
<td>$\delta$</td>
<td>$3C(1 + w)$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

**Note.**—Integral form solutions generally valid on superhorizon scales are presented. We assume $K = 0 = \Lambda$, thus $\alpha = \left[ 1 - \frac{H}{a} \int \frac{r}{a} dt \right]$. Since $p = \rho(0)$, $w$ is not necessarily constant in time. Since these solutions consider only the dominating growing and decaying modes, the vanishing decaying mode simply means that the dominating mode has cancelled. Also, we cannot determine the behavior of leading order of $\Psi$, $\kappa$, and $\epsilon$ terms in the SG. Except for these cases, for constant $w$, one can show that the growing modes match with the ones in Table 2.
equation (67). In the pressureless medium, equation (45), which applies to the SG or the CG cases, can be written as

\[ \delta + 2H\dot{\delta} - 4\pi G\rho \delta = \frac{1}{a^2 H} \left[ a^2 H^2 \left( \frac{1}{H} \delta \right)' \right]' = 0. \quad (66) \]

This equation can simply be solved as

\[ \delta = (k^2 - 3K)CH \int \frac{dt}{a^2}, \quad (67) \]

where \( C \) is an integration constant introduced to match with equation (63) (see eql. [21]); the decaying mode is again absorbed into the lower bound of the integration. Note that both equations (66) and (67) are valid even in a background with general \( K \) and \( \Lambda \).

Thus, although equations (63) and (67) are valid on any scales in MDE, the solutions presented in Table 8 are valid only on superhorizon scales in general.\(^5\)

In this case, comparing Table 8 with Table 5 we may note the following. In the CG (and part of the SG) the integral-form solutions in Table 8 also apply outside Jeans scale. Except for the \( \delta \), the ZSG results in Table 8 also apply outside Jeans scale (\( \varphi \) in this gauge is the one in eq. [63]). In the other gauges the integral-form solutions are valid in general only on superhorizon scales. From Table 5 one can note that depending on whether the scale is larger or smaller than the horizon (\( \propto \sim 1 \)) many of the solutions in MDE change their behavior.

3.7.1. Applications

Consider a case with \( K = \Lambda = 0 \) and \( w = \text{constant} \). Since \( a \propto t^n \) where \( n = 2/[3(1 + w)] \), from equation (63) the growing mode of \( \varphi_x \) behaves as (Lyth 1985)

\[ \varphi_x = \frac{1}{1 + n} C \left[ 1 + \frac{2}{3(1 + w)} \right]^{-1} C. \quad (68) \]

Since \( C \) remains constant even considering changes in \( p(\mu) \), a change in the background equation of state will cause a corresponding change in \( \varphi_x \). The expression in equation (68) is not valid for a sudden jump of \( w \); as can be seen in equation (63) this causes a change in \( \varphi_x \) to occur in a continuous manner even at a sudden alteration of the equation of state, Hwang & Vishniac (1991). This conservation of \( C \) (which can be identified with \( \xi \) and many other GI \( \varphi \) variables on superhorizon scales) and consequent determination of all the variables provides a simple method of deriving the density perturbation spectrum generated from the early inflationary era. As noted in § 3.3, this \( a \)-dependence of \( \varphi_x \) occurs because the dominant growing modes of \( \delta \) and \( \varphi \) have the same order as in equation (44).

Another useful application can be made to a medium composed of both dust (\( p = 0 \)) and radiation (\( p = 1/3 \mu \)) where

\[ \delta + 2H\dot{\delta} - 4\pi G\rho \delta = \frac{1}{a^2 H} \left[ a^2 H^2 \left( \frac{1}{H} \delta \right)' \right]' = 0. \quad (66) \]

For \( K = 0 = \Lambda \) we have

\[ a = y^2 + 2y, \quad y \equiv \sqrt{\frac{\mu \eta}{24}} \eta = (-1 \pm \sqrt{2}) \frac{\eta}{\eta_{rd}}, \quad (69) \]

where the subscript \( \eta_{rd} \) indicate the epoch of equal density of radiation and dust, and \( a(\eta_{rd}) = 1 \). The solutions in some gauges are known in the literature (Kodama & Sasaki 1984; the solutions in the SG were known in Field & Shepley 1968, and Nariai et al. 1967; see also Heath 1978, 1982). In the following we present only the solutions for \( \varphi_x \) and \( \delta \) since the rest of the solutions follow from simple algebraic manipulations. Directly from equation (63), using equation (69) we have

\[ \varphi_x = \frac{C}{(y + 2)^3} \left( \frac{3}{5} y^3 + \frac{18}{5} y^2 + \frac{22}{3} y + \frac{16}{3} \right) + D \frac{y + 1}{y(y + 2)^3}, \quad (70) \]

where \( D \) is a coefficient of the decaying mode. From equation (21) we have

\[ \delta \propto \frac{4k^2}{\mu \eta} \frac{y^2(y + 2)^2}{(y + 1)^3} \varphi_x. \quad (71) \]

In RDE (\( y \ll 1 \)) and MDE (\( y \gg 1 \)) we can recover the corresponding solutions we already derived:

\[ \begin{align*}
\delta \propto & \ a^2, \quad a^{-1} \propto t, \quad t^{-1/2} \ (y \ll 1), \\
\delta \propto & \ a^{-3/2} \propto t^{1/2}, \quad t^{-1} \ (y \gg 1).
\end{align*} \quad (72) \]

This method of using the large-scale integral-form solutions may be widely useful in many situations where we know the evolution of the background scale factor.

3.8. Conservation Variables

From Table 8, we can read

\[ \varphi_{\mu} = \varphi_{\kappa} = \varphi_{\xi} = \varphi(\xi = \zeta) = C, \quad (73) \]

where the dominating decaying modes vanished (we neglect gauge mode in the SG). We can also see that for general \( p(\mu) \), equation (43), which is the case for UCG, is still valid except for \( \chi \).

From equations (8)–(14), for general \( K, \Lambda, \) and \( \sigma \), we can derive

\[ \begin{align*}
\phi_x(\xi = \zeta) & = \frac{1}{\mu + p} \left[ \frac{k^2}{3a^2} \Psi_x - He \right], \\
\phi_{\mu} & = \frac{H}{\mu + p} \left[ \frac{k^2 - 3K}{a^2} \left( -\frac{c_s^2}{4\pi G} \varphi_x + \frac{2}{3} \sigma \right) - e \right] + \frac{K}{a^2} \chi, \\
\phi_\kappa & = \frac{3H}{k^2 - c_s^2} \left[ \frac{k^2 - 3K}{a^2} H(1 + 3c_s^2) \varphi_x + 12\pi GHe \right] + \frac{k^2}{3a^2} \chi, \\
\phi_{\varphi} & = \frac{1}{\mu + p} \left[ \frac{k^2}{a^2} \left( -\frac{4\pi G(1 + 3c_s^2)}{H} \varphi_x + 12\pi GHe \right) \right] dt + \frac{k^2}{3a^2} \chi. \quad (74)
\end{align*} \]

\[ \]

\[ ^6 \] The case of pressureless matter perturbations in a uniform radiation background cannot be treated using equation (63). This case need part of a two-component model. See Guyot & Zeldovich 1970; Mészáros 1974; Groth & Peebles 1975; §12 of Peebles 1980; Rozgacheva & Sunyaev 1981.

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Thus for ideal fluid \((\epsilon = 0 = \sigma)\) and \(K = 0\), all these \(\dot{\phi}\)'s are \([k/(aH)]^2\) order higher than the other variables. Out of these, the case for \(\zeta \equiv \phi_2\) looks simpler than the other cases.

From equations (9), (12), and (13) we can also derive the \(\phi\) equation without taking any gauge condition as

$$\dot{\phi} = \frac{K}{k^2 - 3K} \kappa + H\alpha + \frac{4\pi Gk^2}{k^2 - 3K}\Psi.$$  \hspace{1cm} (75)

Thus for \(K = 0\) we have

$$\dot{\phi} = H\alpha + \frac{4\pi G\Psi}{k^2 - 3K}.$$  \hspace{1cm} (76)

From Table 8 we can note the following. For the CG, the UEG, and the UDG the right-hand side of equation (76) becomes \([k/(aH)]^2\) order higher than \(\phi\) which is equal to \(C\) on superhorizon scales, thus \(\dot{\phi} = 0\) in that order. For the UCG the right-hand side of equation (76) exactly vanishes which must be the case since the UCG takes \(\phi = 0\) as the gauge condition. For the ZSG, all the terms in equation (76) have comparable order and \(\dot{\phi}\) does not vanish on superhorizon scales (also in any scale in MDE).

The existence of a conserved perturbed curvature variable \(\phi\) on superhorizon scales (eq. [73]), especially the fact that \(\phi\) in the CG is conserved even on subhorizon scales in MDE, in combination with the validity of the Friedmann equation (the first of eq. [6]) even to linear order implies the following important conclusion (see H2, HH1): In the perturbed FLRW universe regions separated in the large scale (larger than Jeans length in MDE), evolve as Friedmann-like universes with different but constant curvature (Lyth 1985; Lyth & Mukherjee 1988). We call this as the “large-scale Friedmann-like evolution.” This was proved in covariant manner and also in perturbation analysis in H2 and HH1.

We would like to note that our (the authors) general view about conservation variables have shifted from \(\zeta\) into \(\phi_2\)'s in many gauge choices shown in equation (73). Now, what seems central in understanding the large-scale behavior is the integration constant \(C\) which contains all the information about the spatial structure of the perturbations, and also about initial conditions imprinted when the scale first reached the large scale during early acceleration (inflation) phase of the universe. The \(C\) widely appearing in this paper is the same \(C\). On superhorizon scales \(C\) can be identified with \(\phi\) in many gauge choices where \(\zeta\) is one of them. Moreover, if we determine \(C\) from some initial condition (usually microscopic quantum fluctuations

3 Recently we found a method based on the UCG is most suitable in treating scalar field perturbations. From the analysis based on that gauge we were able to present a simplified but complete picture of the generation of density spectrum from scalar field quantum fluctuations by the inflation mechanism, now including the accompanying metric fluctuations. In a practical sense, what is calculated is the coefficient \(C\) determined from the scalar field vacuum quantum fluctuations which were magnified into macroscopic size fluctuations through the inflation mechanism.

where \(y \equiv \omega^{1/2} - \pi(1 + \beta)/2\) (this excludes the \(w = 0\) case; exact solutions for the \(w = 0\) case were presented in § 3.4). In this scales, by reducing the exact solutions in Table 1 and using

$$f' \simeq -w \int f \, dx \simeq \frac{1}{x} (-\alpha \sin y + \beta \cos y),$$  \hspace{1cm} (78)

which are valid to the order we are considering, one can generate all the rest of the solutions. These solutions are presented in Table 9. The remnant gauge modes in the SG arise from the lower bound of integration of the exact solution in Table 1.

We note that on subhorizon scales the density perturbations of most gauge choices, except for the UDG (in which case we have \(\delta_\phi = -8\pi G\Psi/H\), are coincident and have the value

$$\delta_\phi \simeq \delta \simeq \frac{1}{\sqrt{\omega}} \rho^{-\beta + 1}(\alpha \cos y + \beta \sin y) \simeq \rho^{-\beta + 1}x.$$  \hspace{1cm} (79)

(Again, for \(w = 0\) see Table 5). For \(w > 0\), equation (79) shows oscillatory behavior. The amplitude remains constant for \(w = 1/3\) \((\beta = 1, \text{RDE})\); for exact solution see eqs. [28], [30]. For the negative pressure case (\(w < 0\), the solution behaves in an exponential way. This is natural since we are considering a negative pressure medium. For example, the pressure term with \(c_s^2\) in equation (25) becomes negative, thus acting with the same sign as the gravity. However, this behavior is based on the assumption that \(w = \text{constant}\) which implies \(c_s^2 = w\).

For \(v_i\) and \(\varphi_2\) we have

$$v_i \varphi = \frac{3\beta}{2(1 + \beta)} x^{-\beta + 1}x', \quad \varphi_2 = \frac{3}{2} \beta^2 x^{-\beta + 1}x,$$  \hspace{1cm} (80)

where according to Table 9 both variables simply vanish in the UDG, which simply means that to get equation (80) we need to consider the next-order terms in that gauge. Also notice that, except in the UDG, \(\zeta\) is dominated by the \(\epsilon\) term on subhorizon scales

$$\zeta = \frac{\beta}{2(1 + \beta)} x^{-\beta + 1}x.$$  \hspace{1cm} (81)

On superhorizon scales \(\zeta\) was dominated by \(\varphi\) in most gauge choices (see eq. [41]).

3.10. Additional Comments

In the literature, one can find some common errors surrounding the SG which is caused by confusing the gauge mode with the decaying mode (Adams & Canuto 1975; Press & Vishniac 1980; p. 57 of Kodama & Sasaki 1984; Voglisi 1986; Schönh 1989). This confusion arises from some accidental cancellation of the dominating decaying mode in the large-scale asymptotic expansion, or by incorrectly using the general looking solution in equation (28) and taking the limit in the pressureless medium (compare eqs. [28], [36] with eq. [29]). This has been noted in Kodama & Sasaki 1984; Bednarz (1985; Lyth & Stewart 1990; H3). In fact, in the pressureless medium one can use equation (29) as the general solution and in this case the accidental cancellation which occurred in the general \(w\) case does not occur for \(w = 0\). For a convenient reference, we have summarized the temporal behavior of density perturbations on superhorizon scales in Table 10.

As we see in Table 1, all of the gauge-modes in the SG behave as proportional to \(t^{-3} (\kappa/H)\) has an additional mode with \(t^{-1} + 2(1 + \beta)\) in any scale. In a pressureless medium, however, the decaying modes of \(\delta\) also behave as proportional
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<td>$\gamma \Psi / H$</td>
<td>0</td>
<td>$3\beta x^{-\beta} S$</td>
</tr>
<tr>
<td>$(x / \bar{\rho})^2 H_L$</td>
<td>$\frac{3}{2(1 + \beta)} x^{-\beta + 2} S$</td>
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</tr>
<tr>
<td>$\kappa / H$</td>
<td>$\frac{3}{2(1 + \beta)} x^{-\beta + 2} S$</td>
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</tr>
<tr>
<td>$a$</td>
<td>$-\frac{2 - \beta}{2(1 + \beta)} x^{-\beta + 1} S$</td>
<td>$-\frac{3}{2} \beta x^{-\beta - 1} S$</td>
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<td>$\varphi$</td>
<td>$\frac{3\beta^2}{2(1 + \beta)} x^{-\beta} S$</td>
<td>$\frac{3}{2} \beta^2 x^{-\beta - 1} S$</td>
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<tr>
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<tr>
<td>$(x / \bar{\rho})^2 H_L$</td>
<td>$\frac{3\beta}{2} x^{-\beta} S$</td>
<td>$-\frac{3}{2} \beta + \frac{1}{\beta^2} x^{-\beta + 1} S$</td>
</tr>
<tr>
<td>$\kappa / H$</td>
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<td>$-3\beta x^{-\beta - 1} S$</td>
<td>0</td>
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<tr>
<td>$\varphi$</td>
<td>$\frac{3}{2} \beta^2 x^{-\beta - 1} S + g x^{-\beta - 1}$</td>
<td>$\frac{3}{2} \beta^2 x^{-\beta - 1} S + g x^{-\beta - 1}$</td>
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<td>$-x^{-\beta + 1} S$</td>
</tr>
<tr>
<td>$(x / \bar{\rho})^2 H_L$</td>
<td>$-\frac{3}{2} x^{-\beta + 1} S$</td>
<td>$\frac{1}{2\beta(1 + \beta)} x^{-\beta + 2} S$</td>
</tr>
<tr>
<td>$\kappa / H$</td>
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<td>$\frac{1}{2\beta(1 + \beta)} x^{-\beta + 2} S$</td>
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<tr>
<td>$\varphi$</td>
<td>0</td>
<td>$\frac{1}{2(1 + \beta)} x^{-\beta + 1} S$</td>
</tr>
<tr>
<td>$\delta$</td>
<td>$x^{-\beta + 1} S$</td>
<td>0</td>
</tr>
</tbody>
</table>

Note.—Asymptotic solutions valid on subhorizon scales.

$$\sqrt{wS} \equiv \delta \cos \left( \sqrt{wS} \frac{\pi}{2} (1 + \beta) \right) + \beta \sin \left( \sqrt{wS} \frac{\pi}{2} (1 + \beta) \right).$$

The cases for $w = 0, -1/3 (\beta = 2, \text{ diverge})$ are treated separately, where the exact solutions are derived in Tables 5 and 6, respectively. Note that except in the UDG case, $\delta$ behaves the same way for other gauge choices.
to $t^{-1}$ (see Table 5). This was correctly pointed out by Olson (1976) whose covariant analysis has ended up in calculating density perturbations in the SG (Lyth & Stewart 1990).

Equation (15.10.57) in Weinberg (1973) is an incorrect limit of the large-scale asymptote which results in picking up a gauge mode instead of the physical decaying mode in RDE. The correction was made in equations (10)–(13) of H3. (As noted in § 3.6, the case treated as an inflationary era in H3 using $w = -1$ should be considered as the case where $w$ is very close to $-1$.) The original analysis in Hawking (1966), which is based on the covariant equations using the spatial gradient variable, corresponds to the GI counterpart of the CG analysis. The analysis for the nonzero pressure in that paper was known to contain an algebraic error, and the explicit correction has been made in § IIIb1 of HV1.

4. SUMMARY

We summarize some of the main results presented in this paper.

1. Exact and asymptotic solutions are presented for ideal-fluid medium for six fundamentally different gauge choices. These are: exact solutions for $w = $ constant (Table 1); asymptotic solutions on superhorizon scales for $w = $ constant (Tables 2 and 3) and the case for $w = 1$ (Table 4); exact solutions for $w = 0$ (Table 5); exact solutions for $w = -1/3$ considering general $K$ (Table 6); exact and asymptotic solutions for $w = -1$ (Table 7); integral-form solutions for general $p(u)$ valid on superhorizon scales (Table 8); asymptotic solutions on subhorizon scales for $w = $ constant (Table 9).

2. Conserved quantities on superhorizon scales in each gauge are identified (§§ 3.3.2 and 3.8). In most of the gauge choices it is the perturbed curvature variable $\varphi$ (eq. [73]), which remains constant on superhorizon scales independent of changing background equation of state; an exception occurs for the ZSG where we have equation (63).

3. The coefficient $C$ appears as an integration constant of the growing mode part of many variables in the large scale. Being an integration constant, it remains constant for general background equation of state, and in many gauge choices $\varphi$ is identified with $C$ as in equation (73). Information about spatial structure of the perturbations are all coded in $C$, and this information may have been encoded during the inflation era when the scale first reached the large scale. This initial condition imprinted in $C$ will be preserved afterward and, as our full solutions manifestly show, the information about all the variables can be determined from it (assuming that the decaying modes do not have significant roles).

4. Correspondences with the Newtonian analysis are clarified (§§ 2.4, 3.4.1): $\delta_{\text{ICG}}$, $\varphi_{\text{ZSG}}$, and $\Psi_{\text{ZSG}}$ ($\sim v_1$; see eq. [17]) have correct Newtonian correspondence as the perturbed density, potential, and velocity variables, respectively.

5. Roles of the general integral-form solutions available in the large scale and their wide potential applications are explained (§ 3.7).

6. On subhorizon scales the behavior of $\delta$ in many gauges (except for the UDG) coincide and show correct correspondences with the Newtonian result in the pressureless case.

7. At various levels we tried to relate our results with the ones known in the literature and were able to clarify some of them.

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APPENDIX

DE DONDER GAUGE

Let us introduce

$$\Gamma^a \equiv g^{bc} \Gamma^a_{bc} = \frac{1}{\sqrt{-g}} (\sqrt{-g} g^{ab})_{,b}.$$  \hspace{2cm} (A1)

In Minkowski space, letting $\Gamma^a = 0$ is usually called as de Donder, or harmonic, gauge condition. In the FLRW background, with $g_{ab} \equiv g_{ab} + h_{ab}$ where $h_{ab}$ is based on $g_{ab}$, we have

$$\Gamma^a = \Gamma^a + h^{ab}_{\ ,b} - \frac{1}{2} h^a - h^a \Gamma^a_{bc}.$$  \hspace{2cm} (A2)

Directly from equation (A1), using

$$g \equiv \det g_{ab} = \tilde{g}(1 + h), \quad \tilde{g} = -a^3 \tilde{g}^{(3)}, \quad h \equiv k^a = \alpha + 3\varphi + \nabla^2 \gamma,$$

$$g_{ab} \equiv g_{ab} + h_{ab}.$$

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or from equation (A2) we can derive
\[ \Gamma^0 = \frac{1}{a} \left( 2H - \dot{a} - H \alpha - \kappa \right), \]
\[ \Gamma^a = \frac{1}{a^2} \left[ \Gamma^{(3 \alpha)}(1 - 2\varphi) + a \beta^a + 2a \beta^a - a^a - \varphi^a + (V^2 + 2K)\Gamma^{(3 \alpha)} \right]. \] (A4)

Under the gauge transformation \( \tilde{x}^a = x^a + \xi^a \) we have (§ 2.2 of H1)
\[ \tilde{h}_{ab} = h_{ab} - (\xi_{ac} b + \xi_{bc} a), \quad \tilde{\gamma} = \gamma - \frac{\xi}{a^2}, \quad \tilde{\beta} = \beta - \frac{T}{a} + a \left( \frac{\xi}{a^2} \right), \] (A5)
where \( \xi_a = \xi_{a}(a) \) (thus \( \xi = a^{-2} \xi_{a} \)). Using these one can derive the evolution of the pure gauge mode of \( \Gamma^a \) (eq. [15])
\[ \Gamma_{a \text{ gauge}} = -\xi_{ac} b - \frac{\xi}{a^2} R_{ab} + 2 \xi_{bc} \Gamma_{bc}. \]
Thus
\[ \Gamma_{a \text{ gauge}} = \frac{1}{a} \left[ \tilde{T} + HT - (3H + a^{-2} \nabla^2)T \right], \]
\[ \Gamma_{a \text{ gauge}} = \frac{1}{a^2} \left[ \dot{\xi}^a - 2H \xi^a - \frac{\Delta^2}{a^2} \right] + \left( \frac{V^2 + 2K}{a^2} \right) \xi^a + 2 \frac{1}{a^2} \xi^b \Gamma^{(3 \alpha)} \right] + 2H T \Gamma^{(3 \alpha)} \], (A6)

where \( \nabla^2 \xi = \xi_{ab} \); \( T = a^2 \xi = \xi \). In the cosmological case the de Donder gauge may correspond to imposing \( \Gamma^a = 0 \) (in our time varying background the name "harmonic" may lose its meaning; compare with § 7.4 of Weinberg 1973). Since our set of equations are explicitly GL under a spatial gauge transformation we do not need to impose the spatial component of the gauge condition. In this case the time slicing gauge-mode is incompletely determined, and to remove the gauge mode we need to solve the second-order differential equation for \( T \) (first of eq. [A6]). Thus, with this gauge the equation may end up in a fourth-order differential equation where two of the modes are gauge modes which need to be removed by solving the first of equations (A6).

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