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Cosmological perturbations in generalized gravity theories: conformal transformation

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Abstract. A broad class of generalized gravity theories can be cast into Einstein gravity with a minimally coupled scalar field using a suitable conformal rescaling of the metric. Using this conformal equivalence between the theories, we derived the equations for the background and the perturbations, and the general asymptotic solutions for the perturbations in the generalized gravity from the simple results known in the minimally coupled scalar field. Results for the scalar and tensor perturbations can be presented in unified forms. The large-scale evolutions for both perturbations are characterized by corresponding conserved quantities. The simple result for the scalar perturbation is possible mainly due to our proper choice of a gauge-invariant combination which corresponds to the perturbed scalar field in the uniform-curvature gauge.

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1. Introduction

Studies of generalized forms of Einstein's theory as theories for gravity have been made by many authors. Some of the most studied generalized gravity theories include Brans–Dicke theory, induced gravity, dilaton coupling, non-minimally coupled scalar field, nonlinear scalar curvature coupling, etc. We have a variety of motivations for considering more generalized forms for the gravity: the Mach principle, quantum backreaction, renormalization, higher-dimensional unification, string theory, cosmology, etc. In the context of generalized gravity theories, Einstein gravity can be regarded as a special limiting case.

In [1, 2] we presented a thorough derivation of the equations and general asymptotic solutions describing the scalar and tensor perturbations in the conventional cosmological spacetime supported by generalized gravity. The progress made in [1, 2] was based on discovering the role of a proper choice of the gauge in treating the scalar perturbation; previous studies in [3–5] were made in a different gauge condition. We found that by employing a suitable gauge the scalar perturbation could be described in a simple manner similar to that in Einstein gravity. When we dealt with the minimally coupled scalar field we found that the choice of uniform-curvature gauge, or equivalently a corresponding gauge-invariant combination of variables, simplifies the problem; see [6–9]. In [1, 2] we found that when we deal with generalized gravity involving the scalar field and the scalar curvature, the uniform-curvature gauge again suits the problem. In fact, we discovered that in the large-scale limit, neglecting the transient (decaying) solution, the same solution for the minimally coupled scalar field remains valid over a broad class of generalized gravity theories. The equations and the general solutions for both the scalar and tensor perturbations in a class of generalized gravity theories can be presented in unified forms.

It is known in the literature that using a conformal transformation the class of generalized gravity theories we mentioned can be cast into Einstein gravity with a minimally coupled scalar field [10, 3]. Using the mathematical equivalence between the generalized gravity theories and the simple minimally coupled scalar field under the conformal transformation, without losing any rigour, we can derive the equations and solutions in the generalized gravity directly from the known results in Einstein gravity. In this paper we use the conformal equivalence between the theories only as a mathematical tool; for discussions concerning the physics, see [11]. (The conformal transformation properties of the background and perturbed quantities in the cosmological spacetime are presented in [3, 5]. However, at the time of the work in [3–5] the proper role of the uniform-curvature gauge was not known.) Using the conformal transformation, the rigorously derived results in [1] can be rederived in a considerably simpler manner. In this paper we will present the derivation. We will summarize the results for the individual gravity case in tabular forms.

2. Generalized $f(\phi, R)$ gravity

We consider a general class of gravity theories with the Lagrangian

$$L = \tfrac{1}{2} f(\phi, R) - \tfrac{1}{2}\omega(\phi)\phi^{;a}\phi_{,a} - V(\phi). \tag{1}$$

The gravitational field equation and the equation of motion for the scalar field are

$$G_{ab} = \frac{1}{F}\left[\omega\left(\phi_{,a}\phi_{,b} - \tfrac{1}{2}g_{ab}\phi^{;c}\phi_{,c}\right) - g_{ab}\frac{RF - f + 2V}{2} + F_{,a;b} - g_{ab}F^{;c}{}_{c}\right], \tag{2}$$

$$\phi^{;a}{}_{a} + \frac{1}{2\omega}\left(\omega_{,\phi}\phi^{;a}\phi_{,a} + f_{,\phi} - 2V_{,\phi}\right) = 0. \tag{3}$$

where we have defined $F \equiv \partial f/(\partial R)$. We call it the generalized $f(\phi, R)$ gravity theory. It includes diverse classes of gravity theories as special cases; see table 1.

Table 1. Cases of the generalized $f(\phi, R)$ gravity.

$f(\phi, R)$ gravity	$L = \frac{1}{2} f(\phi, R) - \frac{1}{2}\omega(\phi)\phi^{;a}\phi_{,a} - V(\phi)$	$F = F(\phi, R)$
$f(R)$ gravity	$L = \frac{1}{2} f(R)$	$F = F(R), \quad \phi = 0$
R^2 gravity	$L = \frac{1}{2}\left(R - R^2/6M^2\right)$	$F = 1 - R/3M^2, \quad \phi = 0$
Generalized scalar tensor theory	$L = \phi R - \omega(\phi)\phi^{;a}\phi_{,a}/\phi - V(\phi)$	$F = 2\phi, \quad \omega \to 2\omega(\phi)/\phi$
Brans–Dicke theory	$L = \phi R - \omega\phi^{;a}\phi_{,a}/\phi$	$F = 2\phi, \quad \omega \to 2\omega/\phi, \quad V = 0$
$F(\phi)R$ gravity	$L = \frac{1}{2}F(\phi)R - \frac{1}{2}\omega(\phi)\phi^{;a}\phi_{,a} - V(\phi)$	$F = F(\phi)$
Dilaton gravity	$L = \frac{1}{2}e^{-\phi}\left(R + \phi^{;a}\phi_{,a}\right)$	$F = e^{-\phi}, \quad \omega = -e^{-\phi}, \quad V = 0$
Generally coupled scalar field	$L = \frac{1}{2}\left(\gamma - \xi\phi^2\right)R - \frac{1}{2}\phi^{;a}\phi_{,a} - V(\phi)$	$F = \gamma - \xi\phi^2, \quad \omega = 1$
Nonminimally coupled scalar field	$L = \frac{1}{2}\left(1 - \xi\phi^2\right)R - \frac{1}{2}\phi^{;a}\phi_{,a} - V(\phi)$	$F = 1 - \xi\phi^2, \quad \omega = 1$
Conformal coupling	$L = \frac{1}{2}\left(1 - \frac{1}{6}\phi^2\right)R - \frac{1}{2}\phi^{;a}\phi_{,a} - V(\phi)$	$F = 1 - \frac{1}{6}\phi^2, \quad \omega = 1$
Minimally coupled scalar field	$L = \frac{1}{2}R - \frac{1}{2}\phi^{;a}\phi_{,a} - V(\phi)$	$F = 1, \quad \omega = 1$
Induced gravity	$L = \frac{1}{2}\epsilon\phi^2 R - \frac{1}{2}\phi^{;a}\phi_{,a} - V(\phi)$	$F = \epsilon\phi^2, \quad \omega = 1$

3. Conformal transformation to Einstein gravity

Using the conformal transformation the generalized $f(\phi, R)$ gravity can be transformed into Einstein gravity with an additional scalar field. By the conformal transformation the metric is redefined as

$$\hat{g}_{ab} = \Omega^2 g_{ab}, \tag{4}$$

where Ω is a spacetime position-dependent factor. We use hats to denote quantities based on the conformally transformed metric frame. By defining the conformal factor as

$$\Omega \equiv \sqrt{F} \equiv \mathrm{e}^{\frac{1}{2}\sqrt{\frac{2}{3}}\psi}, \tag{5}$$

where ψ is a new dynamical variable, one can show that (1) can be transformed into (for a derivation, see equations (42), (43) of [3])

$$\hat{L} = \frac{1}{2}\hat{R} - \frac{\omega}{F}\frac{1}{2}\phi^{\hat{;}a}\phi_{\hat{;}a} - \frac{1}{2}\psi^{\hat{;}a}\psi_{\hat{;}a} - \hat{V}(\phi, \psi), \qquad \hat{V}(\phi, \psi) \equiv \frac{RF - f + 2V}{2F^2}. \tag{6}$$

Thus, our original generalized $f(\phi, R)$ gravity is cast into the Einstein theory with an additional scalar field, ψ, with a special potential term $\hat{V}(\phi, \psi)$. In most interesting situations, ϕ and ψ are dependent on each other. *Assuming* $\psi = \psi(\phi)$ and introducing a new scalar field $\hat{\phi}$ with $\hat{\phi} = \hat{\phi}(\phi)$, (6) can be transformed into a Lagrangian for the minimally coupled scalar field $\hat{\phi}$

$$\hat{L} = \tfrac{1}{2}\hat{R} - \tfrac{1}{2}\hat{\phi}^{\hat{;}a}\hat{\phi}_{\hat{;}a} - \hat{V}(\hat{\phi}). \tag{7}$$

For this, $\hat{\phi}$ should satisfy

$$\mathrm{d}\hat{\phi} = \sqrt{\frac{\omega}{F}\,\mathrm{d}\phi^2 + \mathrm{d}\psi^2}. \tag{8}$$

4. Perturbed universe model

As the metric describing the model universe, we consider a spatially homogeneous, isotropic and flat (FLRW) background and general perturbations of the scalar and the tensor types

$$\mathrm{d}s^2 = -(1 + 2\alpha)\,\mathrm{d}t^2 - a\beta_{,\alpha}\,\mathrm{d}t\,\mathrm{d}x^\alpha + a^2\left[g^{(3)}_{\alpha\beta}(1 + 2\varphi) + 2\gamma_{|\alpha\beta} + 2H_T Y^{(t)}_{\alpha\beta}\right]\mathrm{d}x^\alpha\,\mathrm{d}x^\beta. \tag{9}$$

$a(t)$ is the cosmic scale factor. $g^{(3)}_{\alpha\beta}$ is a comoving part of the background 3-space metric and a vertical bar indicates a covariant derivative based on $g^{(3)}_{\alpha\beta}$; in the flat FLRW background we have $g^{(3)}_{\alpha\beta} = \delta_{\alpha\beta}$. $Y^{(t)}_{\alpha\beta}(\boldsymbol{x})$ is a symmetric, trace-free and transverse harmonic function with $\nabla^2 Y^{(t)}_{\alpha\beta} = -k^2 Y^{(t)}_{\alpha\beta}$; ∇^2 is a Laplacian operator based on $g^{(3)}_{\alpha\beta}$. The perturbative order quantities $\alpha(\boldsymbol{x}, t)$, $\beta(\boldsymbol{x}, t)$, $\gamma(\boldsymbol{x}, t)$ and $\varphi(\boldsymbol{x}, t)$ characterize the scalar perturbation, whereas $H_T(\boldsymbol{x}, t)$ characterizes the tensor perturbation. β and γ are affected by the spatial coordinate transformation in the FLRW spacetime. Since the FLRW spacetime is spatially homogeneous and isotropic we can easily avoid using these spatially gauge-dependent variables [12]. A combination $\chi(\boldsymbol{x}, t) \equiv a(\beta + a\dot{\gamma})$ is a variable such that it is spatially gauge invariant. For the scalar field we let $\phi(\boldsymbol{x}, t) = \bar{\phi}(t) + \delta\phi(\boldsymbol{x}, t)$ where an overbar indicates the background quantity; we neglect the overbar unless it is necessary. Now, the variables α, φ, χ and $\delta\phi$ are spatially gauge invariant, but are temporally gauge dependent. H_T is gauge invariant. For the gauge transformation properties, see section 2.2 of [13].

In handling the scalar perturbations in the cosmological background we prefer the *gauge ready method* introduced in [13] which fixes only the spatial gauge condition (by using the spatially gauge-invariant variable χ) and leaves the temporal gauge condition for later convenient use depending on the situation. We have several fundamental temporal gauge conditions: the gauge conditions concerning the metric are $\alpha \equiv 0$ (the synchronous gauge), $\varphi \equiv 0$ (the uniform-curvature gauge), $\chi \equiv 0$ (the zero-shear gauge) and the uniform-expansion gauge which fixes the perturbed part of the extrinsic curvature. There also exist gauge conditions which fix the energy–momentum tensor part, and they are the comoving gauge, the uniform-density gauge, etc. Out of these several choices, except for the synchronous gauge, each of the gauge conditions completely fixes the temporal gauge mode. The variable in any one of these gauge conditions corresponds to a unique gauge-invariant combination which involves the considered variable and the variable used in the gauge condition. The variables under these gauge conditions can be equivalently considered as the gauge-invariant variables. Thus, as long as we are working in these gauge conditions (which exclude the synchronous gauge) we do not need to worry about the remnant gauge mode. Instead, we need to pay attention to choosing the *proper* gauge condition which is most convenient to the problem. According to Bardeen in [12] 'The moral is that one should work in the gauge that is mathematically most convenient for the problem at hand'. This suggestion is implemented in the gauge-ready method proposed in [13]. In [8, 2], after going through all the fundamental gauge conditions by using the gauge-ready method, we have found that the uniform-curvature gauge is most suitable for treating the scalar field and the classes of the generalized gravity theories considered in this paper.

We decompose the conformal factor Ω into the background and the perturbed part as

$$\Omega(\boldsymbol{x}, t) \equiv \bar{\Omega}(t)[1 + \delta\Omega(\boldsymbol{x}, t)]. \tag{10}$$

From equation (5), we have

$$\bar{\Omega} = \sqrt{F} = \mathrm{e}^{\frac{1}{2}\sqrt{\frac{2}{3}}\psi}, \qquad \delta\Omega = \frac{\delta F}{2F} = \frac{1}{2}\sqrt{\frac{2}{3}}\delta\psi. \tag{11}$$

We can show that the only changes under the conformal transformation are the following (see section 4.1 of [3] and section 2.1 of [13]):

$$\hat{a} = a\bar{\Omega}, \qquad \mathrm{d}\hat{t} = \bar{\Omega}\,\mathrm{d}t, \qquad \hat{\alpha} = \alpha + \delta\Omega, \qquad \hat{\varphi} = \varphi + \delta\Omega. \tag{12}$$

Thus, for example, we have ($H \equiv \dot{a}/a$):

$$\hat{H} = \frac{1}{\Omega}\left(H + \frac{\dot{\Omega}}{\Omega}\right), \qquad \hat{\chi} = \Omega\chi. \tag{13}$$

From equation (8) we can show

$$\dot{\hat{\phi}} = \sqrt{\frac{\omega}{F}\dot{\phi}^2 + \frac{3\dot{F}^2}{2F^2}}, \qquad \frac{\delta\hat{\phi}}{\dot{\hat{\phi}}} = \frac{\delta\phi}{\dot{\phi}} = \frac{\delta F}{\dot{F}}. \tag{14}$$

Relations among $\hat{\phi}$, ϕ and F in the individual gravity cases are summarized in table 2.

5. Perturbations in Einstein gravity

We consider a minimally coupled scalar field. The Lagrangian is given in (7). In this section we *neglect* hats on the background and perturbed quantities. The equations describing the

Table 2. Conformally transformed scalar field.

General forms	$\mathrm{d}\hat{\phi} = \sqrt{\frac{\omega}{F}\mathrm{d}\phi^2 + 3\mathrm{d}F^2/2F^2}$	$\dot{\hat{\phi}} = \sqrt{\frac{\omega}{F}\dot{\phi}^2 + \frac{3\dot{F}^2}{2F^2}}$	$\frac{\delta\hat{\phi}}{\dot{\hat{\phi}}} = \frac{\delta\phi}{\dot{\phi}} = \frac{\delta F}{\dot{F}}$	$\hat{V} = \frac{RF-f+2V}{2F^2}$
$f(R)$ gravity, R^2 gravity	$\hat{\phi} = \sqrt{\frac{3}{2}}\ln F$	$\hat{\phi} = \sqrt{\frac{3}{2}}\ln F$	$\delta\hat{\phi} = \sqrt{\frac{3}{2}}\frac{\delta F}{F}$	$\hat{V} = \frac{RF-f}{2F^2}$
Generalized scalar tensor theory	$\hat{\phi} = \int\sqrt{\omega(\phi) + \frac{3}{2}}\frac{\mathrm{d}\phi}{\phi}$	$\dot{\hat{\phi}} = \sqrt{\omega(\phi) + \frac{3}{2}}\frac{\dot{\phi}}{\phi}$	$\frac{\delta\hat{\phi}}{\dot{\hat{\phi}}} = \frac{\delta\phi}{\dot{\phi}}$	$\hat{V} = \frac{V}{4\phi^2}$
Brans–Dicke theory	$\hat{\phi} = \sqrt{\omega + \frac{3}{2}}\ln\phi$	$\hat{\phi} = \sqrt{\omega + \frac{3}{2}}\ln\phi$	$\delta\hat{\phi} = \sqrt{\omega + \frac{3}{2}}\frac{\delta\phi}{\phi}$	$\hat{V} = 0$
$F(\phi)R$ gravity	$\hat{\phi} = \int\sqrt{\frac{\omega}{F} + \frac{3}{2F^2}\left(\frac{\mathrm{d}F}{\mathrm{d}\phi}\right)^2}\mathrm{d}\phi$	$\hat{\phi} = \int\sqrt{\frac{\omega}{F} + \frac{3\dot{F}^2}{2F^2\dot{\phi}^2}}\mathrm{d}\phi$	$\frac{\delta\hat{\phi}}{\dot{\hat{\phi}}} = \frac{\delta\phi}{\dot{\phi}}$	$\hat{V} = \frac{V}{2F^2}$
Dilaton gravity	$\hat{\phi} = \frac{1}{\sqrt{2}}\phi$	$\hat{\phi} = \frac{1}{\sqrt{2}}\phi$	$\delta\hat{\phi} = \frac{1}{\sqrt{2}}\delta\phi$	$\hat{V} = 0$
Generally coupled scalar field	$\hat{\phi} = \int\frac{\sqrt{\gamma+\xi(6\xi-1)\phi^2}}{\gamma-\xi\phi^2}\mathrm{d}\phi$	$\dot{\hat{\phi}} = \frac{\sqrt{\gamma+\xi(6\xi-1)\phi^2}}{\gamma-\xi\phi^2}\dot{\phi}$	$\frac{\delta\hat{\phi}}{\dot{\hat{\phi}}} = \frac{\delta\phi}{\dot{\phi}}$	$\hat{V} = \frac{V}{(\gamma-\xi\phi^2)^2}$
Nonminimally coupled scalar field	$\hat{\phi} = \int\frac{\sqrt{1+\xi(6\xi-1)\phi^2}}{1-\xi\phi^2}\mathrm{d}\phi$	$\dot{\hat{\phi}} = \frac{\sqrt{1+\xi(6\xi-1)\phi^2}}{1-\xi\phi^2}\dot{\phi}$	$\frac{\delta\hat{\phi}}{\dot{\hat{\phi}}} = \frac{\delta\phi}{\dot{\phi}}$	$\hat{V} = \frac{V}{(1-\xi\phi^2)^2}$
Conformal coupling	$\hat{\phi} = \sqrt{6}\tanh^{-1}\frac{\phi}{\sqrt{6}}$	$\hat{\phi} = \sqrt{6}\tanh^{-1}\frac{\phi}{\sqrt{6}}$	$\delta\hat{\phi} = \frac{\delta\phi}{1-\frac{1}{6}\phi^2}$	$\hat{V} = \frac{V}{(1-\frac{1}{6}\phi^2)^2}$
Minimally coupled scalar field	$\hat{\phi} = \phi$	$\hat{\phi} = \phi$	$\delta\hat{\phi} = \delta\phi$	$\hat{V} = V$
Induced gravity	$\hat{\phi} = \sqrt{6 + \frac{1}{\epsilon}}\ln\phi$	$\hat{\phi} = \sqrt{6 + \frac{1}{\epsilon}}\ln\phi$	$\delta\hat{\phi} = \sqrt{6 + \frac{1}{\epsilon}}\frac{\delta\phi}{\phi}$	$\hat{V} = \frac{V}{(\epsilon\phi^2)^2}$

evolution of the background are (see equations (2)–(4) of [6]):

$$H^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V\right), \qquad \dot{H} = -\frac{1}{2}\dot{\phi}^2, \qquad \ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0. \tag{15}$$

The third equation follows from the first two.

When we manage the gravity theory involving the scalar field our experience tells us that the uniform-curvature gauge is the most convenient one; equivalently we can take the gauge-invariant variables with a subindex φ. For the uniform-curvature gauge, see section 3 of [7]. Thorough perturbation analyses for a minimally coupled scalar field in the uniform-curvature gauge were made in [6–9]. An example of the gauge-invariant combination is

$$\delta\phi_\varphi \equiv \delta\phi - \frac{\dot{\phi}}{H}\varphi \equiv -\frac{\dot{\phi}}{H}\varphi_{\delta\phi}. \tag{16}$$

$\delta\phi_\varphi$ becomes $\delta\phi$ in the uniform-curvature gauge which takes $\varphi \equiv 0$ as the gauge condition. The action expanded to the second order in the perturbed scalar field is presented in (39) of [7] as (in [7] we adopted some results from the Lagrangian based analyses in [14])

$$S = \frac{1}{2}\int a^3\left\{\delta\dot{\phi}_\varphi^2 - \frac{1}{a^2}\delta\phi_\varphi{}^{|\alpha}\delta\phi_{\varphi,\alpha} + \frac{H}{a^3\dot{\phi}}\left[a^3\left(\frac{\dot{\phi}}{H}\right)^{\cdot}\right]^{\cdot}\delta\phi_\varphi^2\right\}\sqrt{g^{(3)}}\,\mathrm{d}t\,\mathrm{d}^3x, \tag{17}$$

where in the flat FLRW background we have $g^{(3)} = 1$. A closed-form equation for the *scalar field* perturbation and the large- and small-scale asymptotic solutions are (see equations (7), (22), (12), (16) of [6]):

$$\delta\ddot{\phi}_\varphi + 3H\delta\dot{\phi}_\varphi - \left\{\frac{1}{a^2}\nabla^2 + \frac{H}{a^3\dot{\phi}}\left[a^3\left(\frac{\dot{\phi}}{H}\right)^{\cdot}\right]^{\cdot}\right\}\delta\phi_\varphi = 0, \tag{18}$$

$$\delta\phi_\varphi(\boldsymbol{x}, t) = \frac{\dot\phi}{H}\left[-C(\boldsymbol{x}) + D(\boldsymbol{x})\int^t \frac{H^2}{a^3\dot\phi^2}\,\mathrm{d}t\right], \tag{19}$$

$$\delta\phi_\varphi(\boldsymbol{k}, \eta) = \frac{1}{a}\left[c_1(\boldsymbol{k})\mathrm{e}^{\mathrm{i}k\eta} + c_2(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}k\eta}\right], \tag{20}$$

where we have used $\mathrm{d}\eta \equiv \mathrm{d}t/a$ and $\nabla^2 \to -k^2$. $C(\boldsymbol{x})$ and $D(\boldsymbol{x})$ in the large-scale solution are the coefficients of the growing and decaying solutions, respectively. Quantum field-theoretical analyses of (18) in the context of cosmological curved spacetime can be found in [6, 7, 9]. Solutions in (19) and (20) are valid for a general $V(\phi)$. Using equation (15), equation (18) can be written in a compact form as (see equation (11) of [6])

$$\frac{H}{a^3\dot\phi}\left[\frac{a^3\dot\phi^2}{H^2}\left(\frac{H}{\dot\phi}\delta\phi_\varphi\right)^{\cdot}\right]^{\cdot} - \frac{1}{a^2}\nabla^2\delta\phi_\varphi = 0. \tag{21}$$

The equation and the asymptotic solutions for the *gravitational wave* are (see equation (101) of [13])

$$\ddot H_T + 3H\dot H_T - \frac{1}{a^2}\nabla^2 H_T = 0, \tag{22}$$

$$H_T(\boldsymbol{x}, t) = C_g(\boldsymbol{x}) - D_g(\boldsymbol{x})\int_0^t \frac{1}{a^3}\,\mathrm{d}t, \tag{23}$$

$$H_T(\boldsymbol{k}, \eta) = -\frac{1}{a}\left[c_{g1}(\boldsymbol{k})\mathrm{e}^{\mathrm{i}k\eta} + c_{g2}(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}k\eta}\right]. \tag{24}$$

The action for the gravitational wave can be found in [15] and section 18 of [17].

The *vorticity perturbation* does not directly couple with the scalar-type gravity theory; it will evolve according to the angular momentum conservation in the expanding background (see section 3.2.2 of [3] and [16]).

6. Perturbations in generalized gravity theories

Using equation (12) we can show that the following quantities are invariant under the conformal transformation:

$$\mathrm{d}\eta, \quad \nabla^2, \quad k, \quad \varphi_{\delta\phi}, \quad \frac{H}{\dot\phi}\delta\phi_\varphi, \quad H_T. \tag{25}$$

We regard the quantities in section 5 (equations (15)–(24)), as being in the conformal frame, and thus are hatted. Using the conformal transformation properties in (10)–(14), (25) we can derive the corresponding counterparts in the original frame which are now valid for the individual generalized gravity theory included in the generalized $f(\phi, R)$ gravity.

For the background, equation (15) leads to

$$H^2 = \frac{1}{3F}\left(\frac{\omega}{2}\dot\phi^2 + \frac{RF - f + 2V}{2} - 3H\dot F\right), \tag{26}$$

$$\dot H = -\frac{1}{2F}\left(\omega\dot\phi^2 + \ddot F - H\dot F\right), \tag{27}$$

$$\ddot\phi + 3H\dot\phi + \frac{1}{2\omega}\left(\omega_{,\phi}\dot\phi^2 - f_{,\phi} + 2V_{,\phi}\right) = 0. \tag{28}$$

When we derive (28) we may need $R = 6(2H^2 + \dot H)$; equation (28) also follows from (26) and (27).

The same form of (16) remains valid in the generalized gravity theories. The action in (17) becomes (we ignore the surface terms)

$$S = \frac{1}{2}\int a^3 \, \frac{\omega + 3\dot{F}^2/2\dot{\phi}^2 F}{\left(1 + \dot{F}/2HF\right)^2} \Bigg\{ \delta\dot{\phi}_\varphi^2 - \frac{1}{a^2}\delta\phi_\varphi{}^{|\alpha}\delta\phi_{\varphi,\alpha} + \frac{H}{a^3\dot{\phi}}\frac{\left(1 + \dot{F}/2HF\right)^2}{\omega + 3\dot{F}^2/2\dot{\phi}^2 F}\left[a^3 \frac{\omega + 3\dot{F}^2/2\dot{\phi}^2 F}{\left(1 + \dot{F}/2HF\right)^2}\left(\frac{\dot{\phi}}{H}\right)^{\cdot}\right]^{\cdot} \delta\phi_\varphi^2 \Bigg\} \sqrt{g^{(3)}}\,\mathrm{d}t\,\mathrm{d}^3x. \tag{29}$$

For the scalar perturbation, equations (18)–(20) lead to

$$\delta\ddot{\phi}_\varphi + \left\{3H + \frac{\left(1 + \dot{F}/2HF\right)^2}{\omega + 3\dot{F}^2/2\dot{\phi}^2 F}\left[\frac{\omega + 3\dot{F}^2/2\dot{\phi}^2 F}{\left(1 + \dot{F}/2HF\right)^2}\right]^{\cdot}\right\}\delta\dot{\phi}_\varphi - \left\{\frac{1}{a^2}\nabla^2 + \frac{H}{a^3\dot{\phi}}\frac{\left(1 + \dot{F}/2HF\right)^2}{\omega + 3\dot{F}^2/2\dot{\phi}^2 F}\left[\frac{\omega + 3\dot{F}^2/2\dot{\phi}^2 F}{\left(1 + \dot{F}/2HF\right)^2}a^3\left(\frac{\dot{\phi}}{H}\right)^{\cdot}\right]^{\cdot}\right\}\delta\phi_\varphi = 0, \tag{30}$$

$$\delta\phi_\varphi(\boldsymbol{x}, t) = \frac{\dot{\phi}}{H}\left[-C(\boldsymbol{x}) + D(\boldsymbol{x})\int_0^t \frac{\left(H + \dot{F}/2F\right)^2}{a^3\left(\omega\dot{\phi}^2 + 3\dot{F}^2/2F\right)}\,\mathrm{d}t\right], \tag{31}$$

$$\delta\phi_\varphi(\boldsymbol{k}, \eta) = \frac{1}{a}\frac{1 + \dot{F}/2HF}{\sqrt{\omega + 3\dot{F}^2/2\dot{\phi}^2 F}}\left[c_1(\boldsymbol{k})\mathrm{e}^{\mathrm{i}k\eta} + c_2(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}k\eta}\right]. \tag{32}$$

In order to derive (30) it is much simpler to use (21) which leads to

$$\frac{\dot{\phi}}{H}\frac{\left(H + \dot{F}/2F\right)^2}{a^3\left(\omega\dot{\phi}^2 + 3\dot{F}^2/2F\right)}\left[\frac{a^3\left(\omega\dot{\phi}^2 + 3\dot{F}^2/2F\right)}{\left(H + \dot{F}/2F\right)^2}\left(\frac{H}{\dot{\phi}}\delta\phi_\varphi\right)^{\cdot}\right]^{\cdot} - \frac{1}{a^2}\nabla^2\delta\phi_\varphi = 0. \tag{33}$$

By expanding (33) we get (30). For the scalar perturbation, we use $\delta\phi$ as the representative one; for theories without $\delta\phi$, such as $f(R)$ gravity, the above results remain valid using the relations in (14).

For the gravitational wave, from equations (22)–(24) we have

$$\ddot{H}_T + \left(3H + \frac{\dot{F}}{F}\right)\dot{H}_T - \frac{1}{a^2}\nabla^2 H_T = 0, \tag{34}$$

$$H_T(\boldsymbol{x}, t) = C_g(\boldsymbol{x}) - D_g(\boldsymbol{x})\int_0^t \frac{1}{a^3 F}\,\mathrm{d}t, \tag{35}$$

$$H_T(\boldsymbol{k}, \eta) = -\frac{1}{a\sqrt{F}}\left[c_{g1}(\boldsymbol{k})\mathrm{e}^{\mathrm{i}k\eta} + c_{g2}(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}k\eta}\right]. \tag{36}$$

Thus, we complete our derivation of the results valid in a class of generalized gravity theories directly from the known results in Einstein gravity using the conformal transformation method. The method itself is generally applicable independently of the gauge conditions (e.g. see [3, 5, 17]).

In the limit of Einstein gravity with a minimally coupled scalar field we have $F = 1 = \omega$ and, apparently, equations (26)–(36) reduce to the corresponding equations in section 5. Remarkably, the growing solutions of $\delta\phi_\varphi$ and H_T in (31) and (35) do not involve F or ω. This implies that, for the growing solution in the large-scale limit, the same solutions in Einstein gravity remain valid in the generalized gravity theories. It is also noteworthy that the equations and the general asymptotic solutions presented in this section are valid considering the general $V(\phi)$, $\omega(\phi)$ and $f(\phi, R)$ in unified forms for the classes of generalized gravity theories we have been considering.

7. Comparison with the zero-shear gauge results

The large-scale asymptotic solutions in the zero-shear gauge are derived in [3, 5]; studies in [17, 18] are also based on this gauge condition. From equations (16) and (18) of [5] we have

$$\delta\phi_\chi(\boldsymbol{x}, t) = -C(\boldsymbol{x})\frac{\dot\phi}{aF}\int_0^t aF\,\mathrm{d}t + d(\boldsymbol{x})\frac{\dot\phi}{aF}, \tag{37}$$

$$\varphi_\chi(\boldsymbol{x}, t) = C(\boldsymbol{x})\left(1 - \frac{H}{aF}\int_0^t aF\,\mathrm{d}t\right) + d(\boldsymbol{x})\frac{H}{aF}, \tag{38}$$

where $d(\boldsymbol{x})$ is a coefficient of the decaying solution. The Einstein gravity limits of (38), (37) and the relation through the conformal equivalence can be found easily; the following relations are useful:

$$\frac{\delta\hat\phi_\chi}{\dot{\hat\phi}} = \frac{\delta\phi_\chi}{\dot\phi} = \frac{\delta F_\chi}{\dot F}, \qquad \hat\varphi_\chi = \varphi_\chi + \frac{\delta F_\chi}{2F}. \tag{39}$$

From section 2.2 of [13] we note that $\varphi_\chi \equiv \varphi - H\chi \equiv -H\chi_\varphi$ and $\delta\phi_\chi \equiv \delta\phi - \dot\phi\chi$ are gauge-invariant combinations which become φ and $\delta\phi$, respectively, under the zero-shear gauge which takes $\chi \equiv 0$ as the gauge condition. Thus, from (16), (37), (38) we can derive the solution in the large-scale limit as

$$\delta\phi_\varphi = \delta\phi - \frac{\dot\phi}{H}\varphi = \delta\phi_\chi - \frac{\dot\phi}{H}\varphi_\chi = -\frac{\dot\phi}{H}C(\boldsymbol{x}). \tag{40}$$

Notice that the decaying solution in the zero-shear gauge cancels out in the uniform-curvature gauge. In fact, in equation (109) of [2] we derived

$$D(\boldsymbol{x}) = -2\nabla^2 d(\boldsymbol{x}). \tag{41}$$

Using the conformal transformation the proof of (41) becomes simple. In the case of minimally coupled scalar field, from (8)–(10) of [8] we can derive

$$\frac{1}{a^2}\nabla^2\varphi_\chi = \frac{\dot H}{H}\dot\varphi_{\delta\phi}. \tag{42}$$

From equations (16), (19), (38) we can arrive at (41). Since equation (41) is invariant under the conformal transformation, it remains valid for the generalized gravity theories and thus is proved. Thus, although it looks complicated, the decaying solution in the uniform-curvature gauge is a *higher-order* term in the large-scale expansion compared with the one in the zero-shear gauge. We also note that the growing solution is *simpler* in the uniform-curvature gauge; for example, it does not involve F which characterizes the non-Einsteinian nature of the theory. Furthermore, the small-scale solution in the uniform-curvature gauge (32) is also much simpler than the one in the zero-shear gauge which is presented in [4]. The complicated form of equation for $\delta\phi_\chi$ in the minimally coupled scalar field is presented in (37) of [8] which can be compared with the simple one for $\delta\phi_\varphi$ in (18).

8. Discussions

In this paper we have derived the equations and the general asymptotic solutions characterizing the evolution of the perturbed universe model, which are valid in a wide class of generalized gravity theories. Compared with our previous work in [1, 2], in this paper we adopt the conformal equivalences of the considered generalized gravity theories to the Einstein one, thus much simplifying the derivation of both the evolution equations and the general asymptotic solutions. Straightforward derivations of the results in section 6, without addressing the conformal equivalence with the minimally coupled scalar field, are presented in [2]. The equations and the asymptotic solutions for the scalar and tensor perturbations in section 6 can be written in unified forms (see table 3). We note that growing solutions of $\varphi_{\delta\phi}$ and H_T are conserved in the large-scale limit. Ignoring the transient solutions, from (16), (31), (35) we have

$$\varphi_{\delta\phi}(\boldsymbol{x}, t) = C(\boldsymbol{x}), \qquad H_T(\boldsymbol{x}, t) = C_g(\boldsymbol{x}). \tag{43}$$

Table 3. Unified form expressions for the scalar and tensor perturbations. We have $v \equiv -z\Phi$ and $z \equiv a\sqrt{Q}$. (In the action formulation of the tensor perturbation we need to take account of the two polarization states properly [15]. In this table we ignore this minor complication; for a proper account, see [20].)

Action	$S = \frac{1}{2}\int a^3 Q\left(\dot\Phi^2 - \frac{1}{a^2}\Phi^{\vert\alpha}\Phi_{,\alpha}\right)\sqrt{g^{(3)}}\,\mathrm{d}t\,\mathrm{d}^3x$	$S = \frac{1}{2}\int\left(v'^2 - v^{\vert\alpha}v_{,\alpha} + \frac{z''}{z}v^2\right)\sqrt{g^{(3)}}\,\mathrm{d}\eta\,\mathrm{d}^3x$
Equation	$\ddot\Phi + \left(3H + \frac{\dot Q}{Q}\right)\dot\Phi - \frac{1}{a^2}\nabla^2\Phi = 0$	$v'' - \left(\nabla^2 + \frac{z''}{z}\right)v = 0$
Large scale	$\Phi = C(\boldsymbol{x}) - D(\boldsymbol{x})\int_0^t (a^3Q)^{-1}\,\mathrm{d}t$	$v = -z\left[C(\boldsymbol{x}) - D(\boldsymbol{x})\int_0^\eta z^{-2}\,\mathrm{d}\eta\right]$
Small scale	$\Phi = -(a\sqrt{Q})^{-1}\left[c_1(\boldsymbol{k})\mathrm{e}^{\mathrm{i}k\eta} + c_2(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}k\eta}\right]$	$v = c_1(\boldsymbol{k})\mathrm{e}^{\mathrm{i}k\eta} + c_2(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}k\eta}$
Scalar pert.	$\Phi = \varphi_{\delta\phi}, \quad Q = \frac{\omega\dot\phi^2 + 3\dot F^2/2F}{(H + \dot F/2F)^2}$	$v = -z\varphi_{\delta\phi}, \quad z = a\frac{\sqrt{\omega\dot\phi^2 + 3\dot F^2/2F}}{H + \dot F/2F}$
Tensor pert.	$\Phi = H_T, \quad Q = F$	$v = -zH_T, \quad z = a\sqrt{F}$

Considering our preceding publications, most of the results presented above are not new. The main point of this work is the considerably simpler derivations of the results in [1, 2] using the conformal relations without losing any mathematical rigour. This, in fact, makes us understand why we were able to have the simple and unified formulation which is valid for the class of generalized gravity theories. The action formulation presented in equation (29) is new and will be useful for a quantum field-theoretic treatment of the perturbation theory; for recent applications, see [19, 20]. The conformal transformation properties of the generalized gravity were known in [10, 3]. The applications of the conformal transformation property in the context of the cosmological perturbations were made in [3, 17]; as we have mentioned, the works in [3, 17] are based on the zero-shear gauge which leads to more complicated results compared with our results based on the uniform-curvature gauge (section 7).

We emphasize the special role of the uniform-curvature gauge in the treatment of the generalized gravity with a dilaton field. The uniform-curvature gauge was first introduced in the literature in [13] as one of the fundamental temporal gauge choices; this gauge was not known in the reviews in [21]. As we briefly mentioned in section 4, the uniform-curvature gauge choice completely fixes the temporal gauge mode, thus each variable in the gauge condition corresponds to a uniquely gauge-invariant combination; e.g. $\delta\phi_\varphi$ in (16) is the unique gauge-invariant combination which becomes the scalar field fluctuation in the uniform-curvature gauge which fixes $\varphi \equiv 0$ as the gauge condition. In this sense the variables in the uniform-curvature gauge can be considered as the equivalently gauge-invariant ones. The previous works on this subject in [3–5, 13, 14, 17, 18] are based on the zero-shear gauge condition. The zero-shear gauge condition also fixes the temporal gauge mode completely, thus variables in the gauge are equivalently gauge invariant. However, as compared in section 7, the analyses and the results in the uniform-curvature gauge are much simpler than the ones in the zero-shear gauge. The solutions in the other fundamental gauge conditions are presented in section VI of [2]. It may be worth mentioning that the variable $v(\equiv -z\varphi_{\delta\phi})$ used in table 3 was first introduced by Mukhanov in [14] in the context of Einstein gravity; in the context of generalized gravity, see [17, 18]. (However, the variable v was not recognized as the combination of the scalar field fluctuation and the spatial curvature variable until a recent paper [22] where the uniform-curvature gauge was rediscovered without noticing our works.) The treatments of the cosmological perturbation in some particular generalized gravity considered in [18, 17, 22] are all based on the conformal transformation technique. Recently, inflation scenarios which involve an additional inflaton field or multiple episodes of inflation have become popular. In [23] we find some attempts to calculate the generated density spectra in the scenarios involving the additional scalar field in generalized gravity; all these works are based on the conformal transformation technique. However, the analyses made in [23] are based on a gauge other than the uniform-curvature gauge and may deserve another look with a new perspective using the proper gauge choice.

The generalized gravity theories we have considered in this paper do not include the following types of generalized gravity theories in the Lagrangian: general couplings with the Ricci or the conformal curvature like $R_{ab}R^{ab}$ or $C_{abcd}C^{abcd}$, couplings involving general derivatives of the scalar curvature and the scalar field, etc. These types of theories cannot be related to Einstein gravity through a conformal transformation. Cosmological perturbation analyses in the context of these more generalized gravity theories [16, 24], and quantum field-theoretical counterparts of the classical results presented in this paper, will be addressed in future papers [19].

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