Cosmological gravitational wave in a gravity with quadratic order curvature couplings

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We present a set of equations describing the cosmological gravitational wave in a gravity theory with quadratic order gravitational coupling terms which naturally arise in quantum correction procedures. It is known that the gravitational wave equation in the gravity theories with a general f(R) term in the action leads to a second order differential equation with the only correction factor appearing in the damping term. The case for a $R^{ab}R_{ab}$ term is completely different. The gravitational wave is described by a fourth order differential equation both in time and space. However, curiously, the contributions to the background evolution are qualitatively the same for both terms. [S0556-2821(97)06808-2]

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I. INTRODUCTION

Quantum correction procedures generate higher order curvature coupling terms in the action for Einstein gravity [1]. Such corrections are needed for more realistic understanding of the early stage of the universe where the curvature of the universe was large enough for the quantum effects. It is known that to one-loop order Einstein action is modified by the additional quadratic order curvature coupling terms [1]. In four-dimensional spacetime, we have two such correction terms: $R^2$ and $R^{ab}R_{ab}$ terms. The first term has attracted much attention in the literature in the context of cosmology and the cosmological structure formation theory based on linear gravitational stability analyses. Inflationary stage was first implemented in $R^2$ gravity before it was realized based on the scalar field [2]. Now, it is well known that through a conformal transformation of the metric $R^2$ gravity can be transformed to a minimally coupled scalar field [3]. Thus, despite its quadratic nature, it has been shown that the dynamical equations describing the structures in the cosmological model remain to the second order differential equation both for the scalar type perturbations and for the gravitational wave [4–7]. Such a symmetry does not exist in the $R^{ab}R_{ab}$ term.

In a series of papers we will investigate the roles of the $R^{ab}R_{ab}$ term in the context of the evolution of the cosmological structures. In the present work we will derive the equations for the background and the gravitational wave. The main results are Eqs. (7) and (8). Since the cosmological stability analysis for $R^{ab}R_{ab}$ term is new, we present the necessary details for the derivation in the Appendixes. We take the Misner-Thorner-Wheeler (MTW) [8] convention except for setting $c = 1 = 8 \pi G$.

II. GRAVITY THEORY WITH THE QUADRATIC ORDER QUANTUM CORRECTION TERMS

We considered a gravity theory represented by the action

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}(R+AR^2+BR^{ab}R_{ab})+L_m\right].$$

where $L_m$ is the matter part Lagrangian. The gravitational field equation becomes

$$R^a_b - \frac{1}{2}\delta^a_b R - A H^{(1)a}_b - B H^{(2)a}_b = T^a_b,$$

$$H^{(1)a}_b = 2R^a_b - 2\delta^a_b \Box R + \frac{1}{2} \delta^a_b R^2 - 2R R^a_b,$$

$$H^{(2)a}_b = R^a_b - \Box R^a_b - \frac{1}{2} \delta^a_b \Box R + \frac{1}{2} \delta^a_b R^c R^d_{cd} + 2 R^a_b R^d_{c d}.$$ 

In four-dimensional spacetime, due to the generalized Gauss-Bonnet theorem, contributions from $R^{abcd}R_{abcd}$ term can be expressed in terms of $H^{(1)a}_b$ and $H^{(2)a}_b$ terms. These modifications of Einstein gravity were first introduced by Weyl, Pauli, and Eddington [9]. Not only these terms arise from the one-loop level quantum correction, these are also known to make the theory renormalizable [10].

III. THE COSMOLOGICAL MODEL AND THE GRAVITATIONAL WAVE

As the model describing the background universe we consider a homogeneous, isotropic, and flat model. The general perturbation in this background model can be decomposed into three types: the scalar, vector, and flat model. The general perturbation in this background model can be decomposed into three types: the scalar, vector, and tensor type perturbations. To the linear order, due to the symmetry in the background, these three types of perturbations decouple from each other and evolve independently. In this paper we will consider only the tensor type perturbation which is transverse and tracefree; the other types of perturbations will be considered elsewhere. The metric can be written as

$$ds^2 = -a^2 d\eta^2 + a^2 (\delta_{\alpha\beta} + 2C_{\alpha\beta}) dx^\alpha dx^\beta,$$

where $a(\eta)$ is a cosmic scale factor. $C_{\alpha\beta}(x, \eta)$ is a transverse and tracefree perturbed order tensor variable based on a metric $\delta_{\alpha\beta}$. It represents the gravitational wave with
\(C_\alpha = 0 = C_{\mu,\alpha} \) thus having two independent components which indicate the two polarization states of the gravitational wave. \(C_{\alpha \beta} \) is invariant under the gauge transformation. The inverse metric, the connection, and the curvatures based on the metric in Eq. (3), which are valid to the linear order in \(C_{\alpha \beta} \), are summarized in Appendix A. Appendix A contains all quantities we need for analyzing Eq. (2) under the metric in Eq. (3) considering to the linear order in \(C_{\alpha \beta} \). Appendix B is another presentation using \(t \) as the time variable where \(dt = a d\eta \), \(\eta \) and \(t \) are the conformal time and the background proper time, respectively. A prime and an overdot indicate the time derivatives based on \(\eta \) and \(t \) respectively.

IV. EQUATIONS FOR THE BACKGROUND AND THE GRAVITATIONAL WAVE

Using the quantities presented in Appendix B, Eq. (2) leads to

\[
T^0_\alpha = -3H^2 - 6(B + 3A)(2\bar{H} - H^2 + 6H^2\bar{H}),
\]

(4)

\[
T^a_\alpha = 0 = T^a_0,
\]

(5)

\[
T^\alpha_\beta = -\delta^\alpha_\beta [2\bar{H} + 3H^2 + 2(B + 3A)(2\bar{H} + 12H\bar{H} + 9\bar{H}^2 + 18H^2\bar{H}) + (1 + 2AR)D^\alpha_\beta + 2AR C^\alpha_\beta - B\left(\bar{D}^\alpha_\beta + 3HD^\alpha_\beta - \frac{\Delta}{a^2} (D^\alpha_\beta + 4HC^\alpha_\beta)\right) - 6(\bar{H} + H^2)C^\alpha_\beta - 4\bar{H} a^2 C^\alpha_\beta],
\]

(6)

where \(R \) and \(D^\alpha_\beta \) are given in Eqs. (B2) and (B9). The background parts of Eqs. (4) and (6) become \(T^0_\beta = T^a_\beta = \delta T^a_\beta \)

\[H^2 + 2(B + 3A)(2\bar{H} - H^2 + 6H^2\bar{H}) = -\frac{1}{3} T^0_0,
\]

\[\bar{H} + 2(B + 3A)(\bar{H} + 3H\bar{H} + 6H^2\bar{H}) = \frac{1}{2} \left( T^0_0 - \frac{1}{3} T^a_\alpha \right).
\]

(7)

The second equation follows from the first one; from \(T^a_\alpha = 0 \) we have \(T^0_\alpha = -H(3T^0_0 - \bar{T}^a_\alpha) \). It is remarkable to see that, to the background order, contributions from \(R \bar{R} R_{ab} \) term are qualitatively the same as the ones from \(R^2 \) term. The perturbed part of Eq. (6) becomes

\[
D^\alpha_\beta + 2A(RD^\alpha_\beta + \bar{R} C^\alpha_\beta) - B\left(\bar{D}^\alpha_\beta + 3HD^\alpha_\beta - \frac{\Delta}{a^2} (D^\alpha_\beta + 4HC^\alpha_\beta)\right) - 6[(\bar{H} + H^2)D^\alpha_\beta + (\bar{H} + H^2)C^\alpha_\beta] = \delta T^a_\beta,
\]

(8)

Equation (8) constitutes a fourth order differential equation for \(C^a_\beta(x, t) \) which describes the evolution of the gravitational wave.

V. CASE FOR \(R^2 \) GRAVITY

In the \(R^2 \) gravity we let \(B = 0 \). Using \(F = 1 + 2AR \), Eq. (7) becomes

\[
H^2 = -H \frac{\dot{F}}{F} + \frac{R(F - 1)}{12F} - \frac{1}{3F} \frac{\dot{\alpha}^0}{\alpha^0},
\]

\[
\bar{H} = -\frac{\bar{H} \dot{F}}{2F} + \frac{1}{2F} \left( \frac{\dot{\alpha}^0}{\alpha^0} - \frac{1}{3} \frac{\bar{\alpha}^a}{\bar{\alpha}^a} \right).
\]

(9)

where the integration constants \(c^a_\beta(x) \) and \(d^a_\beta(x) \) indicate the spatial structures in the growing mode and decaying mode, respectively. Apparently, in the large scale limit without the source term we have a general integral form solution

\[
C^a_\beta(x, t) = c^a_\beta(x) + d^a_\beta(x) \int_0^t \frac{dt}{a^2 \bar{F}}.
\]

(11)

VI. DISCUSSIONS

We have derived the equations describing the evolution of the background universe and the gravitational wave in the gravity modified by considering the quadratic order correction terms in curvatures. Such correction terms generically appear as the one-loop quantum correction in the investigation of the quantum aspects of the gravity [1]. It is known that general \(f(R) \) term in the action can be transformed to the Einstein action by a conformal transformation of the metric. The basic reason why we obtain a second order differential equation for the gravitational wave in the \(R^2 \) gravity may be traced to the underlying symmetry under the conformal transformation [4,7]. However, \(R \bar{R} R_{ab} \) term cannot be transformed to the Einstein one.

Equation (7) describes the evolution of the background universe, and Eq. (8) describes the evolution of the gravitational wave under such a background model. Although the contributions to the background model are qualitatively the same for \(R^2 \) and \(R \bar{R} R_{ab} \) terms, contributions to the evolution of the gravitational wave are different. The similar contributions to the background equation look like a nontrivial result; using Eqs. (B2) and (B4) the similarity can be shown in the
action level. We hope to pay more attention to consequences of Eqs. (7) and (8) in future investigations.

Recently the string theories have attracted much attention as the potential candidates for a successful quantum gravity. The generic low energy effective action of the string theories is represented by an action modified by a dilaton field. The equation for the gravitational wave in such a case is described by an equation which is effectively the same as Eq. (10); see [4,12]. However, the stringy quantum correction terms are more general than our action in Eq. (1); it yields coupling terms involving various combinations of the curvatures, the dilaton field, and their derivatives [13]. Similar investigations in such more general gravity theories will be considered in the future.

Results for the scalar and vector type perturbations and applications of our results to the relevant cosmological situations will be presented in future publications.

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APPENDIX A: LINEAR ORDER QUANTITIES

The inverse metric ($0 = \eta$)

$$g^{00} = -\frac{1}{a^2}, \quad g^{0\alpha} = 0, \quad g^{\alpha\beta} = \frac{1}{a^2} (\delta^{\alpha\beta} - 2 C^{\alpha\beta}). \tag{A1}$$

The connections

$$\Gamma^0_{0\alpha} = \frac{a'}{a}, \quad \Gamma^0_{\alpha 0} = 0, \quad \Gamma^0_{\alpha \beta} = \frac{a'}{a} \delta_{\alpha \beta} + C^{\alpha \beta} + 2 \frac{a'}{a} C_{\alpha \beta},$$

$$\Gamma^\alpha_{0 \beta} = \frac{a'}{a} \delta^{\alpha \beta} + C^{\alpha \beta}, \quad \Gamma^\alpha_{\beta \gamma} = C^{\alpha \beta, \gamma} + C^{\alpha \gamma, \beta} - C^{\beta \gamma, \alpha}, \tag{A2}$$

Riemann curvature

$$R^a_{\beta 0 b 0} = 0 = R^0_{\alpha \beta \gamma}, \quad R^0_{\alpha \beta \gamma} = C^\gamma_{\alpha \beta, \gamma} - C^{\gamma \beta, \alpha}, \quad R^a_{\alpha \beta \gamma} = C^{\alpha \beta, \gamma}, \quad R^a_{\beta 0 \gamma} = C^{\alpha \gamma, \beta}, \quad R^a_{\beta 0 \gamma} = C^{\gamma \beta, \alpha}, \quad R^a_{\gamma \beta 0} = C^{\alpha \beta, \gamma}.$$

$$R^a_{\beta \gamma \delta} = \left( \frac{a'}{a} \right)^2 \delta_{\beta \gamma \delta} + C^{\beta \gamma, \delta} + 2 \frac{a'}{a} C^{\beta \gamma, \delta} + \frac{a'}{a} \delta_{\beta \gamma} C^{\alpha \delta} + \delta_{\beta \gamma} C^{\alpha \delta} - C^{\alpha \delta, \beta \gamma} + 2 \frac{a'}{a} \left( \delta_{\beta \gamma} C^{\alpha \delta} - \delta_{\beta \gamma} C^{\alpha \delta} \right) + C^{\alpha \delta, \beta \gamma} - C^{\beta \gamma, \alpha \delta}.$$

Ricci curvature

$$R^0 = \frac{3}{a^2} \left( \frac{a'}{a} \right)^2, \quad R^0_a = 0 = R^a_0,$$

$$R^a = \frac{6}{a^2} \left( \frac{a'}{a} \right)^2 + \frac{2}{a^2} \left( \frac{a'}{a} \right)^2 \delta_{a} + C^{a, \alpha} + 2 \frac{a'}{a} C^{a, \alpha} - \Delta C^{a} \beta \tag{A4}.$$
\[ R_\beta^a = -\frac{1}{a^2} \left( R_{\beta a} + \frac{a'}{a} R_{\beta}' - 4 \delta_{\beta}^a \frac{a'}{a} \left( \frac{a'}{a} \right) ' \right) \]
\[ - \left( \frac{a'}{a} \right)^2 - \frac{1}{a^2} \left[ D_{\beta a} + \frac{a'}{a} D_{\beta} - 2 \left( \frac{a'}{a} \right)^2 D_{\beta}' - \Delta D_{\beta} \right] \]
\[ - 8 \frac{1}{a^4} \left( \frac{a'}{a} \right)' \left( \frac{a'}{a} \right)^2 C_{\beta}' \]

where

\[ D_{\beta} = \frac{1}{a^2} \left( C_{\beta} + \frac{a'}{a} C_{\beta}' - \Delta C_{\beta} \right) \]

and an overbar indicates a background order of the variable.

**APPENDIX B: IN TERMS OF t**

Using \( t \) as the time variable we have (\( H = \dot{a}/a \))

\[ R_0^a = 3(H + H^2), \quad R_0^0 = 0 = R_0^a, \]

\[ R_\beta^a = (H + 3H^2) \delta_{\beta}^a + \ddot{C}_{\beta} + 3H \dot{C}_{\beta} - \frac{\Delta}{a^2} \dot{C}_{\beta}, \]

\[ R = 6(H + 2H^2), \]

\[ G_0^a = -3H^2, \quad G_0^0 = 0 = G_0^a, \]

\[ G_\beta^a = -(2H + 3H^2) \delta_{\beta}^a + \ddot{C}_{\beta} + 3H \dot{C}_{\beta} - \frac{\Delta}{a^2} \dot{C}_{\beta}, \]

\[ R_\beta^0 = 12(H^2 + 3HH^2 + 3H^4), \]

\[ R^d R_{cd0} = -3(H + H^2)(H + 3H^2), \]

\[ R^d R_{cd0} = 0 = R^d R_{cad}, \]

\[ R^d R_{cd\beta} = -(3H^2 + 8HH^2 + 9H^4) \delta_{\beta} - (3H + 2H^2) \ddot{C}_{\beta} \]

\[ - (7H + 6H^2) H \dot{C}_{\beta} + (H + 2H^2) \frac{\Delta}{a^2} \dot{C}_{\beta}, \]

\[ R_0^0 = -\ddot{R}, \quad R_0^0 = 0 = R_0^a, \]

\[ R_{\beta}^a = -(H \delta_{\beta}^a + \ddot{C}_{\beta}) \dot{R}, \]

\[ \Box R = -(\dddot{R} + 3\dot{R} \ddot{R}), \]

\[ \Box R_0^a = -3(\dddot{R} + 5H \ddot{R} + 2H^2 \dot{R}), \]

\[ \Box R_0^0 = 0 = \Box R_0^a, \]

\[ \Box R_\beta^a = -\delta_{\beta}^a (\dddot{R} + 9H \ddot{R} + 6H^2 + 22H^2 \dot{R}) - D_{\beta}^a + 3H D_{\beta}^a \]

\[ - 2H^2 D_{\beta}^a \frac{\Delta}{a^2} D_{\beta}^a - 8HH \dot{C}_{\beta}, \]

\[ D_{\beta}^a = \ddot{C}_{\beta} + 3H \dot{C}_{\beta} - \frac{\Delta}{a^2} \dot{C}_{\beta}. \]