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Relativistic-Newtonian correspondence of the zero pressure but weakly nonlinear cosmology

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Abstract
It is well known that couplings occur among the scalar-, vector- and tensor-type perturbations of the Friedmann world model in the second perturbational order. Here, we *prove* that, except for the gravitational wave contribution, the relativistic zero-pressure irrotational fluid perturbed to second order in a flat Friedmann background *coincides exactly* with the Newtonian result. Since we include the cosmological constant, our results are relevant to currently favoured cosmology. As we prove that the Newtonian hydrodynamic equations are valid in *all* cosmological scales to the second order, our result has an important practical implication that one can now use the large-scale Newtonian numerical simulation more reliably even as the simulation scale approaches and even goes beyond the horizon. That is, our discovery shows that, in the zero-pressure case, except for the gravitational wave contribution, there are no relativistic correction terms even near and beyond the horizon to the second-order perturbation.

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1. Introduction

Historically, the first proper cosmological analysis appeared only after the advent of Einstein's gravity theory in 1917 [1]: the only known preceding cosmologically relevant discussions can be found in Newton's correspondences with Bentley in 1692 [2]. The expanding world model and its linear structures were first studied in the context of Einstein's gravity in the classic studies by Friedmann in 1922 [3] and Lifshitz in 1946 [4], respectively. In an interesting sequence, the much simpler and, in hindsight, more intuitive Newtonian studies followed later by Milne in 1934 [5] and Bonnor in 1957 [6], respectively. According to Ellis 'It is curious that it took so long for these dynamic models to be discovered after the (more complex) general relativity models were known' [7]. This is particularly so, because in the case without pressure the Newtonian results *coincide exactly* with the previously derived relativistic ones for both the background world model and its first (linear) order perturbations. It would be fair to point out, however, that the ordinarily known Newtonian cosmology (both for the Friedmann background and its linear perturbations) is not purely based on Newton's gravity, but is one guided by Einstein's theory [8]. The zero-pressure system with the cosmological constant describes the current stage of our universe and its large-scale structures in the linear stage remarkably well.

Here we show that such a relativistic-Newtonian correspondence continues even to the weakly nonlinear order. As the observed large-scale structures show weakly nonlinear processes, our relativistic result has theoretical as well as practical significance to interpret and analyse such structures properly at the relativistic level. As a consequence, our result implies that even to the weakly nonlinear (second perturbational) order the well-known Newtonian equations can be used in all cosmological scales including the super-horizon scale.

The known equations are

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\varrho - \frac{\text{const}}{a^2} + \frac{\Lambda c^2}{3}, \tag{1}$$

with $\varrho \propto a^{-3}$ for the background [3, 5], and

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\varrho\delta = 0, \tag{2}$$

for the linear-order perturbations [4, 6]. The variable $a(t)$ is the scale factor, and $\delta \equiv \delta\varrho/\varrho$ with ϱ and $\delta\varrho$ the background and perturbed parts of the density field, respectively; Λ is the cosmological constant. The 'const' part is interpreted as the spatial curvature in Einstein's gravity [3, 9] and the total energy in Newton's gravity [5]. Although equation (2) is also valid with general spatial curvature, the relativistic-Newtonian correspondence is somewhat ambiguous in the case with curvature [10]. Therefore, in the following we consider the flat background only. Equation (2) is valid even in the presence of the cosmological constant Λ. We will include the Λ term in our analyses of the weakly nonlinear stage. The above two equations with vanishing spatial curvature describe remarkably well the current expanding stage of our universe and its large-scale structures which are believed to be in the linear stage. In the small scale, however, the structures are apparently in the nonlinear stage, and even in the large scale a weakly nonlinear study is needed. Up until now, such a weakly nonlinear stage has been studied based on Newton's gravity only.

The case with non-vanishing pressure *cannot* be handled in the Newtonian context, especially for the perturbation. In this work, we will show an additional continuation of relativistic-Newtonian correspondences in the zero-pressure medium by showing that, except for the gravitational wave contribution, the relativistic second-order perturbation is described by the *same* set of equations known in the Newtonian system. That is, except for the coupling with the gravitational waves, the Newtonian equations *coincide* exactly with the relativistic ones even to the second order in perturbations.

For relativistic perturbations, due to the covariance of field equations, we have the freedom to fix the spacetime coordinates by choosing some of the metric or energy–momentum variables at our disposal: this is often called the gauge choice. The relativistic-Newtonian correspondence to the linear order was made by properly arranging the equations using various gauge-invariant variables [11, 10]. In the relativistic case δ in equation (2) is, in fact, δ in the temporal comoving gauge condition which also implies the temporal synchronous gauge in our zero-pressure case [4, 11, 12]. In this work we extend such correspondences to the second order.

2. Fully nonlinear equations

We may start from the completely nonlinear and covariant equations; for a complete set of the covariant (1 + 3) equations, see [14]. We consider a zero-pressure fluid with vanishing isotropic pressure and anisotropic stress, thus $\tilde{p} \equiv 0 \equiv \tilde{\pi}_{ab}$. In the energy frame we take $\tilde{q}_a \equiv 0$. Tildes indicate the covariant quantities, and the Greek and Latin indices indicate the space and spacetime indices, respectively. The momentum conservation gives vanishing acceleration vector $\tilde{a}_a$ to all orders; see equation (27) of [13]. The energy conservation and the Raychaudhury equation ($\tilde{G}^\alpha_\alpha - \tilde{G}^0_0$ part of Einstein's equation) give

$$\tilde{\dot{\tilde{\mu}}} + \tilde{\mu}\tilde{\theta} = 0, \tag{3}$$

$$\tilde{\dot{\tilde{\theta}}} + \frac{1}{3}\tilde{\theta}^2 + \tilde{\sigma}^{ab}\tilde{\sigma}_{ab} - \tilde{\omega}^{ab}\tilde{\omega}_{ab} + 4\pi G\tilde{\mu} - \Lambda = 0, \tag{4}$$

see equations (26) and (28) of [13]; $\tilde{\theta} \equiv \tilde{u}^a{}_{;a}$ is an expansion scalar with $\tilde{u}_a$ a fluid 4-vector, $\tilde{\sigma}_{ab}$ and $\tilde{\omega}_{ab}$ are the shear and the rotation tensors, respectively; if $\tilde{u}_\alpha = 0$ we have no rotation of the fluid 4-vector $\tilde{u}_a$. We set $c \equiv 1$. An overdot with tilde is a covariant derivative along the $\tilde{u}_a$ vector, e.g., $\tilde{\dot{\tilde{\mu}}} \equiv \tilde{\mu}_{,a}\tilde{u}^a$. By combining these equations, we have

$$\left(\frac{\tilde{\dot{\tilde{\mu}}}}{\tilde{\mu}}\right)^{\tilde{\cdot}} - \frac{1}{3}\left(\frac{\tilde{\dot{\tilde{\mu}}}}{\tilde{\mu}}\right)^2 - \tilde{\sigma}^{ab}\tilde{\sigma}_{ab} + \tilde{\omega}^{ab}\tilde{\omega}_{ab} - 4\pi G\tilde{\mu} + \Lambda = 0. \tag{5}$$

Equations (3)–(5) are valid to all orders, i.e., these equations are fully nonlinear and still covariant.

We consider the *scalar*- and *tensor-type* perturbations in the *flat* Friedmann background without pressure. We ignore the vector-type perturbation; the vector-type perturbation (rotation) is supposed to be unimportant because it always decays in the expanding phase even to the second order, see section VII.E of [13]. We will take the *comoving* gauge, and by ignoring the vector-type perturbations, we have no rotation. In this case the 4-vector becomes $\tilde{u}_\alpha = 0$, thus coinciding with the normal 4-vector $\tilde{n}_a$. We lose no generality by imposing the gauge condition. In our case the energy–momentum tensor becomes $\tilde{T}^0_0 = -\tilde{\mu}$, and $\tilde{T}^0_\alpha = 0 = \tilde{T}^\alpha_\beta$ where $\tilde{\mu}$ is the energy density. We emphasize that as our comoving gauge condition fixes the gauge degree of freedom completely, all variables in this gauge condition are equivalently gauge invariant to the second order: this is in the sense that each of the variables has a unique corresponding gauge-invariant combination, see [13].

In [13] the equations are presented in the normal frame $\tilde{n}_a$ with $\tilde{n}_\alpha \equiv 0$. The fluid 4-vector $\tilde{u}_a$, in general, differs from the normal 4-vector $\tilde{n}_a$. *Only* in the comoving gauge without rotation the two frames coincide. Since the fluid quantities are defined in the fluid ($\tilde{u}_a$) frame, the zero-pressure condition should be imposed in $\tilde{u}_a$ frame. Thus, for the fluid quantities defined in the normal frame, the physical zero-pressure condition implies vanishing pressures (both isotropic and anisotropic) only in the comoving gauge without rotation. In this normal frame, the gauge transformation to the second order causes pressure terms to appear in other gauges, see [13]. In the energy frame, which takes vanishing flux $\tilde{q}_a \equiv 0$ as the frame condition, the comoving gauge condition takes $\tilde{u}_\alpha \equiv 0$ for the fluid 4-vector; here, we ignore the vector-type perturbation. Since $\tilde{u}_\alpha = 0$, it coincides with the normal frame vector. Now, in the normal frame, which takes $\tilde{n}_\alpha \equiv 0$ as the frame condition, the comoving gauge condition without rotation implies $\tilde{q}_a \equiv 0$. Thus, as long as we take the comoving gauge without rotation, in either frame we have $\tilde{q}_a \equiv 0$ and $\tilde{u}_\alpha = 0 = \tilde{n}_\alpha$; i.e., the fluid 4-vector coincides with the normal 4-vector.

3. Correspondence to the second order: a proof

Now, we consider equations perturbed to the second order in the metric and matter variables. We introduce

$$\tilde{\mu} \equiv \mu + \delta\mu, \qquad \tilde{\theta} \equiv 3\frac{\dot{a}}{a} + \delta\theta, \tag{6}$$

where μ and $\delta\mu$ are the background and perturbed energy density, respectively, and $\delta\theta$ is the perturbed part of the expansion scalar; we set $\delta \equiv \delta\mu/\mu$. We *identify* $\mu \equiv \varrho$ to the background order, and

$$\delta\mu \equiv \delta\varrho, \qquad \delta\theta \equiv \frac{1}{a}\nabla \cdot \mathbf{u}, \tag{7}$$

to both the linear- and second-order perturbations. Now, to the second order, after some algebra using perturbed-order quantities presented in [13], the perturbed parts of equations (3) and (4) give (see the appendix)

$$\dot{\delta} + \frac{1}{a}\nabla \cdot \mathbf{u} = -\frac{1}{a}\nabla \cdot (\delta\mathbf{u}), \tag{8}$$

$$\frac{1}{a}\nabla \cdot \left(\dot{\mathbf{u}} + \frac{\dot{a}}{a}\mathbf{u}\right) + 4\pi G\mu\delta = -\frac{1}{a^2}\nabla \cdot (\mathbf{u} \cdot \nabla\mathbf{u}) - \dot{C}^{(t)\alpha\beta}\left(\frac{2}{a}\nabla_\alpha u_\beta + \dot{C}^{(t)}_{\alpha\beta}\right), \tag{9}$$

where $C^{(t)}_{\alpha\beta}$ is the transverse and tracefree tensor-type perturbation (the gravitational waves) introduced in equation (A.1); the indices of $C^{(t)}_{\alpha\beta}$ are raised and lowered by $\delta_{\alpha\beta}$; its contribution in equation (9) comes from the shear term in equation (4); see the appendix. By combining these equations, we have

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta = -\frac{1}{a^2}\frac{\partial}{\partial t}\left[a\nabla \cdot (\delta\mathbf{u})\right] + \frac{1}{a^2}\nabla \cdot (\mathbf{u} \cdot \nabla\mathbf{u}) + \dot{C}^{(t)\alpha\beta}\left(\frac{2}{a}\nabla_\alpha u_\beta + \dot{C}^{(t)}_{\alpha\beta}\right), \tag{10}$$

which also follows from equation (5). Equations (8)–(10) are our extension of equation (2) to the second-order perturbations in Einstein's theory. We will show that, except for the gravitational wave contribution, exactly the same equations also follow from Newton's theory.

The presence of linear-order gravitational waves can generate the second-order scalar-type perturbation by generating the shear terms. The coupling between the scalar-type perturbation and the gravitational waves to the second order was noticed in the original study of the second-order perturbations by Tomita in 1967 [15]. Here, we note the behaviour of the gravitational waves in the linear regime. To linear order the gravitational waves are described by the well-known wave equation [4]

$$\ddot{C}^{(t)}_{\alpha\beta} + 3\frac{\dot{a}}{a}\dot{C}^{(t)}_{\alpha\beta} - \frac{\Delta}{a^2}C^{(t)}_{\alpha\beta} = 0. \tag{11}$$

In the super-horizon scale the non-transient mode of $C^{(t)}_{\alpha\beta}$ remains constant, thus $\dot{C}^{(t)}_{\alpha\beta} = 0$, and in the sub-horizon scale, the oscillatory $C^{(t)}_{\alpha\beta}$ redshifts away, thus $C^{(t)}_{\alpha\beta} \propto a^{-1}$. Note that only time derivatives of $C^{(t)}_{\alpha\beta}$ generate the scalar-type perturbation. Thus, we anticipate that the contribution of gravitational waves to the scalar-type perturbation is not significant to the second order. The quadratic combinations of linear-order scalar-type perturbation can also work as sources for the gravitational waves to the second order [16].

In the Newtonian context, the mass conservation the momentum conservation and Poisson's equation give [17]

$$\dot{\delta} + \frac{1}{a}\nabla \cdot \mathbf{u} = -\frac{1}{a}\nabla \cdot (\delta \mathbf{u}), \tag{12}$$

$$\dot{\mathbf{u}} + \frac{\dot{a}}{a}\mathbf{u} + \frac{1}{a}\nabla \delta\Phi = -\frac{1}{a}\mathbf{u} \cdot \nabla \mathbf{u}, \tag{13}$$

$$\frac{1}{a^2}\nabla^2 \delta\Phi = 4\pi G \varrho \delta, \tag{14}$$

where $\delta\Phi$ is the perturbed gravitational potential. We note that these equations are valid to fully nonlinear order. Equation (8) follows from equation (12), and equation (9) ignoring the gravitational waves follows from equations (13) and (14). Thus, equation (10) also naturally follows in Newton's theory [17]. In this Newtonian situation $\mathbf{u}$ is the perturbed velocity and $\delta \equiv \delta\varrho/\varrho$. This completes our proof of the relativistic-Newtonian correspondence to the second order. Although we successfully identified the relativistic density and velocity perturbation variables, we do *not* have a relativistic variable which corresponds to the Newtonian potential $\delta\Phi$ to the second order. This situation could be understood because the gravitational potential introduced in Poisson's equation reveals the action-at-a-distance nature and the static nature of Newton's gravity theory compared with Einstein's gravity.

Note that in the Newtonian context equations (8) and (9), thus equation (10) as well, without the gravitational waves, are valid to fully nonlinear order. This has an important implication that any non-vanishing third- and higher-order correction terms in the relativistic context should be regarded as purely relativistic effects [18]. To our knowledge the Newtonian equations in equations (12)–(14) were first presented by Peebles in [17].

4. Discussions

We have shown that to the second order, except for the gravitational wave contribution, the zero-pressure relativistic cosmological perturbation equations can be exactly identified with the known equations in Newton's theory. As a consequence, to the second order, we identified the correct relativistic variables which can be interpreted as density $\delta\mu$ and velocity $\delta\theta$ perturbations in equation (7), and we showed that to that order the Newtonian hydrodynamic equations remain valid in all cosmological scales including the super-horizon scale. Our results, showing the equivalence to the second order of the zero-pressure relativistic scalar-type perturbation and the Newtonian ones, may not be entirely surprising considering Birkhoff's theorem [19]; for cosmology related discussions of the theorem, see [17]. However, our results should not be so obvious either, because our system lacks any spatial symmetry contrary to Birkhoff's theorem which is concerned with the spherically symmetric system. It might be as well that our relativistic results give relativistic correction terms appearing to the second order which become important as we approach and go beyond the horizon scale. Our results show that there are *no* such correction terms appearing to the second order, and the correspondence is *exact* to that order. A complementary result, showing the relativistic-Newtonian correspondence in the Newtonian limit of the post-Newtonian approach ($\frac{v}{c}$-expansion with $\frac{GM}{Rc^2} \sim \frac{v^2}{c^2} \ll 1$, thus valid far inside the horizon), can be found in [20]. In fact, the Newtonian hydrodynamic equations naturally appear in the zeroth post-Newtonian order of Einstein's gravity [21]; for the cosmological extension, see [22].

In a classic study of the cosmic microwave background radiation anisotropy in 1967, Sachs and Wolfe have mentioned that 'the linear perturbations are so surprisingly simple that a perturbation analysis accurate to second order may be feasible using the methods of Hawking (1966)' [23]. Our proof of the exact relativistic-Newtonian correspondence to the second order could be regarded as one of such accurate results anticipated in [23]. Indeed, in this work we used the method of Hawking which is the covariant (1+3) equations [24]; for other proofs, see [16].

As we consider a flat background, the ordinary Fourier analysis can be used to study the mode-couplings as in the Newtonian case [25]. Our equations include the cosmological constant, thus compatible with current observations of the large-scale structure and the cosmic microwave background radiation which favour a nearly flat Friedmann world model with non-vanishing Λ [26]. Our result may also have the following important practical cosmological implication. As we have proved that the Newtonian hydrodynamic equations are valid in *all* cosmological scales to the second order, our result has an important cosmological implication that one can use the large-scale Newtonian numerical simulation more reliably in the general relativistic context even as the simulation scale approaches near (and goes beyond) the horizon scale. The fluctuations near the horizon scale are supposed to be linear or weakly nonlinear; otherwise, it is difficult to imagine the presence of spatially homogeneous and isotropic background world model which is the basic assumption and the backbone of modern cosmology.

Since the Newtonian system is exact to second order in nonlinearity, any non-vanishing third- and higher-order perturbation terms in the relativistic analysis can be regarded as the pure relativistic correction. Expanding the fully nonlinear equations in (3)–(5) to third and higher order will give the potential correction terms. For our recent work in the third-order perturbations, see [18]. In [18] we derive the non-vanishing third-order terms which are purely relativistic correction terms. We also show that these correction terms are independent of the horizon and are smaller than the second-order Newtonian/relativistic terms by a factor 10^{-5}, thus negligible indeed.

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Appendix

Since equations (8) and (9) allow us to conclude about the relativistic-Newtonian correspondence, in the following we will derive these equations in some detail using the basic quantities presented in [13]. As the metric we take

$$\mathrm{d}s^2 = -a^2(1+2\alpha)\,\mathrm{d}\eta^2 - 2a\chi_{,\alpha}\,\mathrm{d}\eta\,\mathrm{d}x^\alpha + a^2\left[(1+2\varphi)\delta_{\alpha\beta} + 2C^{(t)}_{\alpha\beta}\right]\mathrm{d}x^\alpha\,\mathrm{d}x^\beta, \tag{A.1}$$

which follows from our convention in equations (49), (175) and (178) of [13]. Here, α, χ and φ are spacetime-dependent perturbed-order variables; $C^{(t)}_{\alpha\beta}$ is a transverse-tracefree tensor-type perturbation variable. In our metric we took the spatial C-gauge which removes the spatial gauge modes completely, thus all the remaining variables we are using are spatially gauge-invariant, see section VI.B.2 of [13]; to the linear order, our notation coincides with Bardeen's in [27]. We work in the temporal comoving gauge which takes $v = 0$ in [13]. The momentum conservation gives $\tilde{a}_\alpha = 0$ which gives $\alpha = -\frac{1}{2a^2}\chi^{,\alpha}\chi_{,\alpha}$, see equation (69) of [13]. In our comoving gauge condition, the 4-vector in equation (53) of [13], using equation (175) in that paper, becomes

$$\tilde{u}_0 = -a, \qquad \tilde{u}_\alpha = 0; \qquad \tilde{u}^0 = \frac{1}{a}, \qquad \tilde{u}^\alpha = \frac{1}{a^2}\chi^{,\beta}\left[(1-2\varphi)\delta^\alpha_\beta - 2C^{(t)\alpha}{}_\beta\right]. \tag{A.2}$$

With these, we have

$$\begin{aligned}\dot{\tilde{\tilde{\mu}}} &= \tilde{\mu}_{,0}\tilde{u}^0 + \tilde{\mu}_{,\alpha}\tilde{u}^\alpha = [\mu(1+\delta)]^{\cdot} + \frac{1}{a^2}\mu\delta_{,\alpha}\chi^{,\alpha}, \\ \dot{\tilde{\tilde{\theta}}} &= \left(3\frac{\dot{a}}{a} + \frac{1}{a}\nabla\cdot\mathbf{u}\right)^{\cdot} + \frac{1}{a^3}(\nabla\cdot\mathbf{u})_{,\alpha}\chi^{,\alpha}.\end{aligned} \tag{A.3}$$

Our $\delta\theta$ is the same as $-\kappa$ in [13]. Using equations (55), (57) and (70) of [13], we can show

$$\tilde{\sigma}^{ab}\tilde{\sigma}_{ab} = \frac{1}{a^4}\left[\chi^{,\alpha\beta}\chi_{,\alpha\beta} - \frac{1}{3}(\Delta\chi)^2\right] + \dot{C}^{(t)\alpha\beta}\left(\frac{2}{a^2}\chi_{,\alpha\beta} + \dot{C}^{(t)}_{\alpha\beta}\right). \tag{A.4}$$

Now, we have to relate χ to our notation. Apparently, we need χ only to the linear order. The $\tilde{G}^0_\alpha$-component of Einstein's equation in equation (197) of [13] gives $\frac{\Delta}{a^2}\chi = -\kappa \equiv \delta\theta \equiv \frac{1}{a}\nabla\cdot\mathbf{u}$. As our $\mathbf{u}$ is of the potential type, i.e., of the form $\mathbf{u} \equiv u_{,\alpha}$, we have $\mathbf{u} = \frac{1}{a}\nabla\chi$ to the linear order. Using these, equations (3) and (4) give

$$\left(\frac{\dot{\mu}}{\mu} + 3\frac{\dot{a}}{a}\right)(1+\delta) + \dot{\delta} + \frac{1}{a}\nabla\cdot\mathbf{u} = -\frac{1}{a}\nabla\cdot(\delta\mathbf{u}), \tag{A.5}$$

$$3\frac{\ddot{a}}{a} + 4\pi G\mu - \Lambda + \frac{1}{a}\nabla\cdot\left(\dot{\mathbf{u}} + \frac{\dot{a}}{a}\mathbf{u}\right) + 4\pi G\mu\delta = -\frac{1}{a^2}\nabla(\mathbf{u}\cdot\nabla\mathbf{u}) - \dot{C}^{(t)\alpha\beta}\left(\frac{2}{a}u_{\alpha,\beta} + \dot{C}^{(t)}_{\alpha\beta}\right). \tag{A.6}$$

To the background order, we have $\dot{\mu} + 3(\dot{a}/a)\mu = 0$ and $3\ddot{a}/a + 4\pi G\mu - \Lambda = 0$; after an integration we recover equation (1) with Λ; in this case the 'const' is an integration constant which can be interpreted as the spatial curvature. The perturbed parts give equations (8) and (9).

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