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Newtonian versus relativistic nonlinear cosmology

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Abstract Both for the background world model and its linear perturbations Newtonian cosmology coincides with the zero-pressure limits of relativistic cosmology. However, such successes in Newtonian cosmology are not purely based on Newton's gravity, but are rather guided ones by previously known results in Einstein's theory. The action-at-a-distance nature of Newton's gravity requires further verification from Einstein's theory for its use in the large-scale nonlinear regimes. We study the domain of validity of the Newtonian cosmology by investigating weakly nonlinear regimes in relativistic cosmology assuming a zero-pressure and irrotational fluid. We show that, first, if we ignore the coupling with gravitational waves the Newtonian cosmology is exactly valid even to the second order in perturbation. Second, the pure relativistic correction terms start appearing from the third order. Third, the correction terms are independent of the horizon scale and are quite small in the large-scale near the horizon. These conclusions are based on our special (and proper) choice of variables and gauge conditions. In a complementary situation where the system is weakly relativistic but fully nonlinear (thus, far inside the horizon) we can employ the post-Newtonian approximation. We also show that in the large-scale structures the post-Newtonian effects are quite small. As a consequence, now we can rely on the Newtonian gravity in analyzing the evolution of nonlinear large-scale structures even near the horizon volume.

Keywords Newtonian cosmology · Relativistic cosmology · Large-scale structure · Nonlinear perturbations

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1 Introduction

In order to interpret results from Einstein's gravity theory *properly* we often need corresponding results in Newton's theory. On the other hand, in order to use results from Newton's gravity theory *reliably* we need confirmation from Einstein's theory. The observed large-scale structures show nonlinear processes are working. Currently, studies of such structures are mainly based on Newtonian physics in both analytical and numerical approaches. One may admit its incompleteness as the simulation scale becomes large because, first, Newton's gravity is an action-at-a-distance, i.e., the gravitational influence propagates instantaneously thus violating causality. Second, Newton's theory is ignorant of the presence of horizon where the relativistic effects are supposed to dominate. One other reason we may add is that Einstein's gravity apparently has quite different structure from Newton's one. The causality of gravitational interactions and consequent presence of the horizon in cosmology are naturally taken into account in the relativistic gravity theory.

In the literature, however, independently of such possible shortcomings of Newton's gravity in the cosmological situation, the physical size of Newtonian simulation, in fact, has already reached the Hubble horizon scale. Common excuses often made by people working in this active field of large-scale numerical simulation are, first, in the small scale one may rely on Newton's theory and, second, as the scale becomes large the large-scale distribution of galaxies looks homogeneous. *If* the deviation from homogeneity is small (linear) Einstein's gravity gives *the same* result as the Newtonian one. Presence of large-scale homogeneity, although difficult to verify observationally, is in fact a crucially important assumption in currently popular cosmology. In order to have proper confirmation, however, we still need to investigate Einstein's case in the nonlinear or weakly nonlinear situations. While the general relativistic cosmological simulation is currently not available, in this work, we will shed light on the situation by a perturbative study of the nonlinear regimes assuming zero-pressure and irrotational fluid in Einstein's gravity. This allows us to investigate the similarity and difference between the two gravity theories in the weakly nonlinear regimes in cosmological situation. We will show that even to the second order in perturbations, except for the coupling with gravitational waves, Einstein's gravity gives *the same* results known in Newton's theory and the pure relativistic corrections appearing in the third order perturbations are independent of the horizon and are small. We also present a complementary approach using the post-Newtonian approximation which can handle weakly relativistic (thus, far inside the horizon) but fully nonlinear situation. We show that the first-order post-Newtonian corrections are again quite small. Thus, now our relativistic analysis *assures* that Newton's gravity is practically reliable even in the weakly nonlinear regimes in cosmology. We set $c \equiv 1$.

2 Nonlinear equations

We start from the completely nonlinear and covariant equations [1]. We need the energy conservation equation and the Raychaudhuri equation. In a zero-pressure

medium without rotation we have [1]

$$\tilde{\dot{\mu}} + \tilde{\mu}\tilde{\theta} = 0, \quad \tilde{\dot{\theta}} + \frac{1}{3}\tilde{\theta}^2 + \tilde{\sigma}^{ab}\tilde{\sigma}_{ab} + 4\pi G\tilde{\mu} - \Lambda = 0, \quad (1)$$

where Λ is the cosmological constant; $\tilde{\mu}$ is the energy density, $\tilde{\theta} \equiv \tilde{u}^a{}_{;a}$ is the expansion scalar with \tilde{u}_a the fluid four-vector, and $\tilde{\sigma}_{ab}$ is the shear tensor; $\tilde{\dot{\mu}} \equiv \tilde{\mu}_{;a}\tilde{u}^a$ is the covariant derivatives along \tilde{u}^a . Tildes indicate the covariant quantities. By combining these equations we have

$$\left(\frac{\tilde{\dot{\mu}}}{\tilde{\mu}}\right)^{\tilde{\cdot}} - \frac{1}{3}\left(\frac{\tilde{\dot{\mu}}}{\tilde{\mu}}\right)^2 - \tilde{\sigma}^{ab}\tilde{\sigma}_{ab} - 4\pi G\tilde{\mu} + \Lambda = 0. \quad (2)$$

These equations are *fully nonlinear* and covariant. To the second and higher order perturbations we also need the momentum constraint part of Einstein's equation.

As the metric we take

$$ds^2 = -a^2(1 + 2\alpha)d\eta^2 - 2a^2\beta_{,\alpha}d\eta dx^\alpha + a^2[g_{\alpha\beta}^{(3)}(1 + 2\varphi) + 2\gamma_{,\alpha|\beta}]dx^\alpha dx^\beta, \quad (3)$$

where α , β , γ and φ are spacetime dependent perturbed-order variables. Spatial indices of perturbed order variables are based on $g_{\alpha\beta}^{(3)}$, and a vertical bar indicates the covariant derivative based on $g_{\alpha\beta}^{(3)}$. We ignored the transverse vector-type perturbation and transverse-tracefree tensor-type perturbation variables. In this perturbation study we will consider the *scalar-type* perturbations up to third order in the *flat* Friedmann background without pressure. The vector-type perturbation has only a decaying solution in expanding medium. The presence of gravitational waves will cause couplings with the scalar-type perturbation to the second and higher orders in perturbations, see [2, 3]. The presence of gravitational waves can be regarded as a pure relativistic effect even to the linear order.

3 Background world model

To the background order, we have $\tilde{\mu} = \mu$ and $\tilde{\theta} = 3\frac{\dot{a}}{a}$ where $a(t)$ is the scale factor, and an overdot indicates the time derivative based on the background proper-time t . Equation (1) gives

$$\dot{\mu} + 3\frac{\dot{a}}{a}\mu = 0, \quad 3\frac{\ddot{a}}{a} + 4\pi G\mu - \Lambda = 0. \quad (4)$$

This was first derived based on Einstein's gravity by Friedmann in 1922 [4], and the Newtonian study followed later by Milne and McCrea in 1934 [5]. In the Newtonian context μ can be identified with the mass density ρ .

4 Linear-order perturbations

To the linear-order perturbations in the metric and energy-momentum variables, we introduce

$$\tilde{\mu} \equiv \mu + \delta\mu, \quad \tilde{\theta} \equiv 3\frac{\dot{a}}{a} + \delta\theta. \quad (5)$$

To the linear order we *identify*

$$\delta\mu \equiv \delta\rho, \quad \delta\theta \equiv \frac{1}{a}\nabla \cdot \mathbf{u}, \quad (6)$$

where $\delta\rho$ and \mathbf{u} are the perturbed mass density and the peculiar velocity in Newtonian context. In all our relativistic (nonlinear) perturbation analyses we take the temporal comoving gauge (which together with the irrotational condition gives $\tilde{u}_\alpha = 0$ for the fluid four-vector \tilde{u}_a) and the spatial $\gamma = 0$ gauge [6]. As these gauge conditions fix the gauge modes completely, all the remaining variables are equivalently gauge-invariant to *all* orders in perturbations; for technical details, see Sect. 5.3 of [6] where an explicitly gauge-invariant combination $\delta\mu_v$ which is the same as the $\delta\mu$ in the temporal comoving gauge and the spatial $\gamma = 0$ gauge can be found in Eq. (282). These gauge conditions and our choice of the perturbation variables are crucially important to make our conclusions.

To the linear order the perturbed part of Eq. (2) gives

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta = 0, \quad (7)$$

which is the well known density perturbation equation in both relativistic and Newtonian contexts; we set $\delta \equiv \delta\mu/\mu$. This equation was first derived based on Einstein's gravity by Lifshitz in 1946 [7], and the Newtonian study followed later by Bonnor in 1957 [8].

It is curious to notice that in both the expanding world model and its linear structures the first studies were made in the context of Einstein's gravity [4, 7], and the much simpler and, in hindsight, more intuitive Newtonian studies followed later [5, 8]. Perhaps these historical developments reflect that people did not have confidence in using Newton's gravity in cosmology before the result was already known in, and the method was ushered by, Einstein's gravity. It may be also true that only after having a Newtonian counterpart we could understand better what the often arcane relativistic analysis shows. It would be fair to point out, however, that the ordinarily known Newtonian cosmology (both for the background world model and its linear perturbations) is not purely based on Newton's gravity, but is a guided one by Einstein's theory [9]. In the cosmological context Newtonian gravity is known to be incomplete and inconsistent; these are due to lack of boundary condition at spatial infinity and the action-at-a-distance nature of Newton's gravity. For the second-order perturbations, currently we only have the Newtonian result known in the literature. Thus, the result only known in Newton's gravity still awaits confirmation from Einstein's theory. Here, we are going to fill the gap by presenting the much needed relativistic confirmation to the second order and the pure general relativistic corrections start appearing from the third order [2, 3].

5 Second-order perturbations

Even to the second order we introduce perturbations as in Eq. (5), and *take* the same identifications made in Eq. (6). To the second order the perturbed part of Eq. (2) gives [2, 6]

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta = -\frac{1}{a^2}\frac{\partial}{\partial t}[a\nabla \cdot (\delta\mathbf{u})] + \frac{1}{a^2}\nabla \cdot (\mathbf{u} \cdot \nabla\mathbf{u}), \quad (8)$$

where the second-order terms are in the right hand side. *Exactly the same* equation also follows from Newton's theory [10]. Although we identified the relativistic density and velocity perturbation variables we *cannot* identify a relativistic variable which corresponds to the Newtonian gravitational potential to the second order [2]. This may not be surprising because Poisson's equation indeed reveals the action-at-a-distance nature and the static nature of Newton's gravity theory compared with Einstein's gravity. In the Newtonian context Eq. (8) is valid to *fully nonlinear* order.

6 Third-order perturbations

Since the zero-pressure Newtonian system is exact to the second order in nonlinearity, all non-vanishing third and higher order perturbation terms in the relativistic analysis can be regarded as the pure relativistic corrections. We use the same *identification* made in Eq. (6) to be valid even to the *third order*, and will *take* the consequent additional third order terms as the pure relativistic corrections. To the third order the perturbed part of Eq. (2) gives [3]

$$\begin{aligned} \ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta &= -\frac{1}{a^2}\frac{\partial}{\partial t}[a\nabla \cdot (\delta\mathbf{u})] + \frac{1}{a^2}\nabla \cdot (\mathbf{u} \cdot \nabla\mathbf{u}) \\ &+ \frac{1}{a^2}\frac{\partial}{\partial t}\{a[2\varphi\mathbf{u} - \nabla(\Delta^{-1}X)] \cdot \nabla\delta\} - \frac{4}{a^2}\nabla \cdot \left[\varphi \left(\mathbf{u} \cdot \nabla\mathbf{u} - \frac{1}{3}\mathbf{u}\nabla \cdot \mathbf{u} \right) \right] \\ &+ \frac{2}{3a^2}\varphi\mathbf{u} \cdot \nabla(\nabla \cdot \mathbf{u}) + \frac{\Delta}{a^2}[\mathbf{u} \cdot \nabla(\Delta^{-1}X)] - \frac{1}{a^2}\mathbf{u} \cdot \nabla X - \frac{2}{3a^2}X\nabla \cdot \mathbf{u}, \end{aligned} \quad (9)$$

where the last two lines are pure third-order terms with

$$X \equiv 2\varphi\nabla \cdot \mathbf{u} - \mathbf{u} \cdot \nabla\varphi + \frac{3}{2}\Delta^{-1}\nabla \cdot [\mathbf{u} \cdot \nabla(\nabla\varphi) + \mathbf{u}\Delta\varphi].$$

This extends Eq. (8) to the third order. Notice that we need the behavior of φ to the linear order only; φ is a perturbed part of three-space metric in Eq. (3), related to the perturbed three-space curvature (in our comoving gauge), and dimensionless. The third-order correction terms in Eq. (9) reveal that all of them are simply of φ -order higher than the second-order terms. Thus, the pure general relativistic effects are at least φ -order higher than the relativistic/Newtonian ones in the second order. To the linear order we have [11]

$$\dot{\varphi} = 0, \quad (10)$$

thus $\varphi = C(\mathbf{x})$ with *no* decaying mode. To the linear order our φ is related to the perturbed Newtonian potential $\delta\Phi$ and the Newtonian peculiar velocity \mathbf{u} as [2, 3]

$$\varphi = -\delta\Phi + \dot{a}\Delta^{-1}\nabla \cdot \mathbf{u}. \quad (11)$$

The temperature anisotropy of cosmic microwave background radiation gives [12, 13]

$$\frac{\delta T}{T} \sim \frac{1}{5}\varphi \sim 10^{-5}. \quad (12)$$

Thus, $\varphi \sim 5 \times 10^{-5}$ in the large-scale limit near horizon scale. Therefore, to the third order, the pure relativistic corrections are *independent* of the horizon scale and depend on the strength of linear order curvature perturbation φ *only*, and are small.

7 Discussion

In this work we show that Newtonian cosmological perturbation equations remain valid in *all* cosmological scales including the super-horizon scale to the second order. We assumed a zero-pressure irrotational fluid and ignored the coupling with gravitational waves. The pure general relativistic correction terms start appearing from the third order. The third order correction terms involve only φ which is independent of the horizon scale and is small in the large scale limit near horizon. Therefore, one can now use the large-scale Newtonian numerical simulation more reliably as the simulation scale approaches and even goes beyond the horizon. All our results include the cosmological constant thus relevant in currently favoured cosmology.

The referee has raised a couple of interesting observations that our conclusions do not refer to the averaging procedure [14], and the pure relativistic corrections start appearing at third order do not depend on the physical scale and on the averaging procedure. Indeed, our relativistic-Newtonian correspondence and the pure relativistic correction terms do not depend on scales nor on averaging procedure. We have reached our conclusions by *comparing* the exact Newtonian equations with the relativistic ones perturbed to the second and third orders *without* taking any averaging procedure. Thus, our relativistic-Newtonian correspondence to the second order and pure relativistic correction terms to the third order are independent of the averaging procedure. Notice, however, that we have achieved our result by choosing special (and proper) variables in certain (spatial and temporal) gauge conditions where all the variables have corresponding unique gauge-invariant combination of variables.

The independence of the third order pure relativistic correction terms from the scale (compared with the second-order terms) is a sure surprise of our result. However, we would like to point out that our pure relativistic correction terms in Eq. (9), certainly depend on our identification of the relativistic gauge-invariant combination of variables as the Newtonian ones to the third order made in Eq. (6); this point was emphasized above Eq. (9). Thus, if we take other identification (of the relativistic variables and gauges) as the Newtonian ones we could end up with correction terms which differ from our result. Based on our successful

and clear identification with exact relativistic-Newtonian correspondence to the second order we believe (therefore, *propose*) the same identification to be valid to the third order, and *suggest* the third order correction terms in Eq. (9) as the pure general relativistic effects (based on our identification of the variables).

The roles of tensor-type perturbation (gravitational waves) are studied in [2, 3]; vector-type perturbation (rotation) is not important because it always decays in the expanding phase. Why Newtonian cosmology, despite its action-at-a-distance nature, still gives the same relativistic results even to the second-order perturbation in *all* scales, leaves room for further clarification. Also, it would be interesting to find cosmological situations where the pure general relativistic correction terms in Eq. (9) could have observationally distinguishable consequences.

Consistency of the Newtonian (nonlinear) cosmology with the Newtonian limit of the post-Newtonian approximation of general relativity was also reported in [15]. In fact, it is well known that the Newtonian hydrodynamic equations naturally appear in the zeroth post-Newtonian order of Einstein's gravity [16]. In [15] it was shown that it is essential to keep the magnetic part of Weyl tensor in order to properly recover even the Newtonian limit in the post-Newtonian approach. In making our proof of the relativistic-Newtonian correspondence to the second order we assumed irrotational and zero-pressure conditions but have not imposed any condition on the magnetic part of the Weyl tensor; for a study based on the covariant equations ignoring the latter quantity, see [17]. In fact, the magnetic part of Weyl tensor does not vanish even to the linear order in perturbations: this quantity valid to the second order is presented in Eq. (96) of [6].

Our nonlinear perturbation approach is applicable to fully relativistic regimes including the super-horizon scales and the early universe. However, it is limited to the weakly nonlinear situations where the nonlinearity is supposed to be small. A complementary approach in handling the large-scale nonlinear evolution in Einstein's gravity is the post-Newtonian approximation. The post-Newtonian approach assumes v/c -expansion with $GM/(Rc^2) \sim v^2/c^2 \ll 1$. Whereas our perturbation approach is applicable in fully relativistic regime assuming weak nonlinearity, the post-Newtonian approach is applicable in fully nonlinear regime assuming weak (relativistic) gravity and slow motion. Thus, whereas the perturbation approach is applicable in all scales assuming weak nonlinearity, the post-Newtonian approach is applicable to fully nonlinear stage but only inside the horizon. Therefore, these two approaches are complimentary in the research of large-scale cosmic structures. Recently, we have extended Chandrasekhar's first-order post-Newtonian hydrodynamic approximation [16] to cosmological situation [18]. In [18] we show that the first-order post-Newtonian correction terms are of order

$$\frac{GM}{Rc^2} \sim \frac{v^2}{c^2} \sim 10^{-6} - 10^{-4}, \quad (13)$$

compared with the Newtonian terms. Thus, although there could appear secular effects due to time-delayed propagation of gravity, the relativistic corrections are quite negligibly small similarly as our third-order pure relativistic correction terms in the weakly nonlinear regime.

Therefore, our weakly nonlinear perturbation study and the fully nonlinear post-Newtonian study assure that in the current stage of the large-scale structure the Newtonian hydrodynamic equations are quite sufficient and reliable in

handling the *dynamics*. However, since we have *not* identified the relativistic variable which corresponds to Newtonian potential to the second order, the Newtonian equations are not supposed to be reliable where the gravitational potential has an important role, like the gravitational lensing.

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