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Why Newtonian gravity is reliable in large-scale cosmological simulations

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ABSTRACT

Until now, it has been common to use Newtonian gravity to study the non-linear clustering properties of large-scale structures. Without confirmation from Einstein's theory, however, it has been unclear whether we can rely on the analysis (e.g. near the horizon scale). In this work we will provide confirmation of the use of Newtonian gravity in cosmology, based on the relativistic analysis of weakly non-linear situations to third order in perturbations. We will show that, except for the gravitational-wave contribution, the relativistic zero-pressure fluid equations perturbed to second order in a flat Friedmann background coincide *exactly* with the Newtonian results. We will also present the pure relativistic correction terms appearing in the third order. The third-order correction terms show that these terms are the linear-order curvature perturbation times the second-order relativistic/Newtonian terms. Thus, the pure general relativistic corrections in the third order are independent of the horizon scale and are small when considering the large-scale structure of the Universe because of the low-level temperature anisotropy of the cosmic microwave background radiation. Since we include the cosmological constant, our results are relevant to currently favoured cosmology. As we prove that the Newtonian hydrodynamic equations are valid in all cosmological scales to second order, and that the third-order correction terms are small, our result has the important practical implication that one can now use the large-scale Newtonian numerical simulation more reliably as the simulation scale approaches and even goes beyond the horizon. In a complementary situation, where the system is weakly relativistic (i.e. far inside the horizon) but fully non-linear, we can employ the post-Newtonian approximation. We also show that in large-scale structures, the post-Newtonian effects are quite small.

Key words: gravitation – hydrodynamics – relativity – cosmology: theory – large-scale structure of Universe.

1 INTRODUCTION

In order to interpret results from Einstein's theory of gravity properly, we often need corresponding results from Newton's theory. On the other hand, in order to use results from Newtonian gravity theory *reliably*, we need confirmation from Einstein's theory. The observed large-scale structures show that non-linear processes are at work; see the 2dF Galaxy Redshift Survey final data release (Colless et al. 2003) and the SDSS data release 3 (Abazajian et al. 2004). Currently, studies of such structures are mainly based on Newtonian physics in both analytical and numerical approaches (for reviews, see Sahni & Coles 1995; Bertschinger 1998; Bernardeau et al. 2002; Cooray & Sheth 2002). One must admit the incompleteness of this approach as the simulation scale becomes larger because, first, Newtonian gravity is an action-at-a-distance, i.e. the gravitational influence propagates instantaneously thus violating causality. Secondly, Newton's theory is ignorant of the presence of the horizon where the relativistic effects are supposed to dominate: near the horizon we have $GM/(\lambda c^2) \sim \lambda^2/\lambda_H^2 \sim 1$ where $\lambda_H \sim c/H$ is the dynamic horizon scale with H as Hubble's constant. Finally, Einstein's gravity apparently has a structure quite different from that of Newtonian gravity. The causality of gravitational interactions and the consequent presence of the horizon in cosmology are naturally taken into account in the relativistic gravity theories, of which Einstein's gravity is the prime example. In this work we will present the similarities and differences between the two gravitational theories in weakly non-linear regimes in cosmological situations.

In the literature, however, independent of any such shortcomings of Newtonian gravity in the cosmological situation, the sizes of Newtonian simulations have already reached the Hubble horizon scale (Colberg et al. 2000; Jenkins et al. 2001; Evrard et al. 2002; Bode & Ostriker 2003; Dubinski et al. 2003; Park et al. 2005). Common explanations often given by people

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working in the active field of large-scale numerical simulation are (i) that in the small scale one may rely on Newton's theory, and (ii) that as the scale becomes larger, the large-scale distribution of galaxies looks linear, in which case Einstein's gravity gives *the same* result as Newtonian gravity. In such a situation, in order to have proper confirmation we still need to investigate the Einstein case in non-linear or weakly non-linear situations. While the general relativistic cosmological simulation is currently not available, in this work we will shed light on the situation by a perturbative study of the non-linear regimes using Einstein's theory of gravity. We will show that even to second order in perturbations, except for coupling to gravitational waves, Einstein's gravity gives *the same* equations known in Newton's theory, and the pure relativistic corrections appearing in the third-order perturbations are independent of the horizon and are small. Thus, our relativistic analysis now provides assurance that Newton's gravity is reliable in practice even in the weakly non-linear regimes in cosmology. Such a comforting conclusion comes from a thorough relativistic analysis of the weakly non-linear situations to third order in perturbations. Despite our simply expressed final conclusion, the results still look surprising and important. We set $c \equiv 1$.

2 FULLY NON-LINEAR EQUATIONS AND PERTURBATIONS

We start from the completely non-linear and covariant (1+3) equations (Ehlers 1961; Ellis 1971, 1973). We need the energy conservation equation and the Raychaudhury equation. For a zero-pressure medium in the energy frame, we have (Noh & Hwang 2004)

$$\tilde{\dot{\tilde{\mu}}} + \tilde{\mu}\tilde{\theta} = 0, \tag{1}$$

$$\tilde{\dot{\tilde{\theta}}} + \frac{1}{3}\tilde{\theta}^2 + \tilde{\sigma}^{ab}\tilde{\sigma}_{ab} - \tilde{\omega}^{ab}\tilde{\omega}_{ab} + 4\pi G\tilde{\mu} - \Lambda = 0, \tag{2}$$

where Λ is the cosmological constant; $\tilde{\theta} \equiv \tilde{u}^a{}_{;a}$ is the expansion scalar, and $\tilde{\sigma}_{ab}$ and $\tilde{\omega}_{ab}$ are the shear and the rotation tensors, respectively. Tildes indicate the covariant quantities based on the space–time metric $\tilde{g}_{ab}$. We have

$$\tilde{\dot{\tilde{\mu}}} \equiv \tilde{\mu}_{,a}\tilde{u}^a$$

and

$$\tilde{\dot{\tilde{\theta}}} \equiv \tilde{\theta}_{,a}\tilde{u}^a$$

which are the covariant derivatives along $\tilde{u}^a$. From these we have

$$\left(\frac{\tilde{\dot{\tilde{\mu}}}}{\tilde{\mu}}\right)^{\tilde{\cdot}} - \frac{1}{3}\left(\frac{\tilde{\dot{\tilde{\mu}}}}{\tilde{\mu}}\right)^2 - \tilde{\sigma}^{ab}\tilde{\sigma}_{ab} + \tilde{\omega}^{ab}\tilde{\omega}_{ab} - 4\pi G\tilde{\mu} + \Lambda = 0. \tag{3}$$

Equations (1)–(3) are valid to all orders, i.e. these equations are fully non-linear and still covariant. Equations (1)–(3) are not complete: to second- and higher-order perturbations, we will need other equations in Einstein's theory.

We consider the scalar- and tensor-type perturbations in the Friedmann background without pressure; we ignore the vector-type perturbation (rotation) because it always decays in the expanding phase even to second order (Noh & Hwang 2004). As the metric we take

$$\mathrm{d}s^2 = -(1+2\alpha)\,\mathrm{d}t^2 - 2a\beta_{,\alpha}\,\mathrm{d}t\,\mathrm{d}x^\alpha + a^2\left[g^{(3)}_{\alpha\beta}(1+2\varphi) + 2\gamma_{,\alpha|\beta} + 2C^{(t)}_{\alpha\beta}\right]\mathrm{d}x^\alpha\,\mathrm{d}x^\beta, \tag{4}$$

where $a(t)$ is the scalefactor; α, β, γ and φ are space–time dependent scalar-type perturbed-order variables; $C^{(t)}_{\alpha\beta}$ is the transverse and trace-free tensor-type metric perturbation (gravitational waves). We take the metric convention in Bardeen (1988) extended to third order (Noh & Hwang 2004). The Greek and Latin indices indicate the space and space–time indices, respectively; the spatial indices of perturbed-order variables are raised and lowered by $g^{(3)}_{\alpha\beta}$ which becomes $\delta_{\alpha\beta}$ if we take Cartesian coordinates in the flat Friedmann background. A vertical bar indicates a covariant derivative based on $g^{(3)}_{\alpha\beta}$. We will take $\gamma \equiv 0$ as the spatial gauge condition which makes all the remaining variables spatially gauge-invariant to all orders of perturbations (Noh & Hwang 2004).

The fluid quantities are ordinarily defined based on the fluid four-vector $\tilde{u}_a$ in the energy frame. Our comoving gauge condition takes vanishing flux $\tilde{q}_a \equiv 0$ (the energy-frame condition), and $\tilde{u}_\alpha \equiv 0$ for the fluid four-vector; here, we ignored the vector-type perturbation. Thus, the fluid four-vector coincides with the normal-frame four-vector $\tilde{n}_a$ which has $\tilde{n}_\alpha \equiv 0$. The condition $\tilde{u}_\alpha = 0$ implies vanishing rotation tensor $\tilde{\omega}_{ab} = 0$. We lose no generality by imposing the gauge condition. Since the comoving gauge condition fixes the temporal gauge mode completely, the remaining variables under this gauge condition are equivalently gauge-invariant; this is true to all orders of perturbations (Noh & Hwang 2004). In our gauge condition the energy–momentum tensor of a zero-pressure irrotational fluid becomes

$$\tilde{T}^0_0 = -\tilde{\mu}, \quad \tilde{T}^0_\alpha = 0 = \tilde{T}^\alpha_\beta, \tag{5}$$

where $\tilde{\mu}$ is the energy density.

3 BACKGROUND AND LINEAR PERTURBATIONS

To the background order, we have $\tilde{\mu} = \mu$ and $\tilde{\theta} = 3(\dot{a}/a)$ where an overdot indicates a time derivative based on t. Equations (1) and (2) give

$$\dot{\mu} + 3\frac{\dot{a}}{a}\mu = 0, \tag{6}$$

$$3\frac{\ddot{a}}{a} + 4\pi G\mu - \Lambda = 0. \tag{7}$$

Combining these equations we have the Friedmann equation

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\mu - \frac{\text{constant}}{a^2} + \frac{\Lambda}{3}, \tag{8}$$

with $\mu \propto a^{-3}$. This equation was first derived based on Einstein's gravity by Friedmann (1922, 1924) and Robertson (1929), and Newtonian study followed later by Milne (1934) and McCrea & Milne (1934). In the Newtonian context, the relativistic energy density μ can be identified with the mass density ϱ. The 'constant' is interpreted as the spatial curvature (K) in Einstein's gravity (Friedmann 1922, 1924) and the total energy in Newton's gravity (McCrea & Milne 1934).

To linear-order perturbations in the metric and energy–momentum variables, we introduce

$$\tilde{\mu} \equiv \mu + \delta\mu, \quad \tilde{\theta} \equiv 3\frac{\dot{a}}{a} + \delta\theta, \tag{9}$$

where μ and $\delta\mu$ are the background and perturbed energy density, respectively, and $\delta\theta$ is the perturbed part of the expansion scalar. We emphasize that our spatial $\gamma = 0$ gauge and temporal comoving gauge conditions defined above equation (5) fix the spatial and temporal gauge degrees of freedom completely. Thus, all variables in these gauge conditions are equivalently gauge-invariant to linear order, i.e. each of the variables has a unique corresponding gauge-invariant combination (Bardeen 1988; Hwang 1991). It is important to point out that the above two statements are valid even in second- and all higher-order perturbations – see section VI in Noh & Hwang (2004). To background order we already identified $\mu \equiv \varrho$. Now, to linear order we identify

$$\delta\mu \equiv \delta\varrho, \quad \delta\theta \equiv \frac{1}{a}\nabla \cdot \boldsymbol{u}. \tag{10}$$

To linear order the perturbed parts of equations (1) and (2) give

$$\dot{\delta} + \frac{1}{a}\nabla \cdot \boldsymbol{u} = 0, \tag{11}$$

$$\frac{1}{a}\nabla \cdot \left(\dot{\boldsymbol{u}} + \frac{\dot{a}}{a}\boldsymbol{u}\right) + 4\pi G\mu\delta = 0. \tag{12}$$

Combining these equations we have

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta = 0, \tag{13}$$

which is the well-known density perturbation equation in both relativistic and Newtonian contexts; we set $\delta \equiv \delta\mu/\mu$. This equation was first derived based on Einstein's gravity by Lifshitz (1946), and Newtonian study followed later by Bonnor (1957). Notice that the relativistic result is *identical* to the Newtonian result. The gravitational wave perturbation present in the relativistic theory simply decouples from the density perturbation and follows the wave equation (Lifshitz 1946)

$$\ddot{C}^{(t)}_{\alpha\beta} + 3\frac{\dot{a}}{a}\dot{C}^{(t)}_{\alpha\beta} - \frac{\Delta - 2K}{a^2}C^{(t)}_{\alpha\beta} = 0, \tag{14}$$

where K is the sign of the background spatial curvature.

It is curious to note that in both the expanding world model and its linear structures, the first studies were made in the context of Einstein's gravity (Friedmann 1922; Lifshitz 1946), and the much simpler and, in hindsight, more intuitive Newtonian studies followed later (Milne 1934; Bonnor 1957). Perhaps these historical developments reflect the fact that people did not have confidence in using Newton's gravity in cosmology before the result was already known, and the method ushered in, using Einstein's gravity. This is also reflected in the historical development of modern cosmology which began only after the advent of Einstein's gravitational theory (Einstein 1917). Furthermore, it is known in the literature that the results in Newtonian cosmology are, in fact, guided by previously known relativistic results – i.e. without the guidance of the relativistic analyses, Newtonian theory could have led to other results (Layzer 1954; Lemons 1988). It may also be true that only after having a Newtonian counterpart could we understand what the often arcane relativistic analysis shows. For the second-order perturbations, however, the history is different from the two previous cases. Currently only the Newtonian result is known in the literature. Thus, the result only known from using Newton's gravitational theory still awaits confirmation from Einstein's theory. Here, we are going to fill the gap by presenting the much-needed relativistic confirmation to second order, and the pure relativistic corrections start appearing from third order.

Although equation (13) is also valid with general spatial curvature, in the following we consider the *flat* background only. As we include the cosmological constant Λ, however, our zero-pressure background and perturbations describe remarkably well the current expanding stage of our Universe and its large-scale structures (Spergel et al. 2003; Tegmark et al. 2004), which are believed to be in the near-linear stage. On small scales, however, the structures are apparently in the non-linear stage, and even on large scales, a study is needed of the weakly non-linear stage. Until now, the weakly non-linear stage has been studied using Newtonian gravity only. In the following we plan to investigate whether such usage of Newtonian gravity in handling the large-scale structure can be justified from the relativistic standpoint by studying the relativistic behaviours of higher-order perturbations.

4 SECOND-ORDER PERTURBATIONS AND NEWTONIAN CORRESPONDENCE

We now consider equations perturbed to second order in the metric and the energy–momentum tensor. Even to second order, we introduce perturbations [as in equation (9)] which are always allowed. We will also *take* the same identifications made in equation (10); this point will be justified by our results below. To second order, the perturbed parts of equations (1) and (2) give (Hwang & Noh 2005a)

$$\dot{\delta} + \frac{1}{a}\nabla \cdot \boldsymbol{u} = -\frac{1}{a}\nabla \cdot (\delta\boldsymbol{u}), \tag{15}$$

$$\frac{1}{a}\nabla \cdot \left(\dot{\boldsymbol{u}} + \frac{\dot{a}}{a}\boldsymbol{u}\right) + 4\pi G\mu\delta = -\frac{1}{a^2}\nabla \cdot (\boldsymbol{u} \cdot \nabla\boldsymbol{u}) - \dot{C}^{(t)\alpha\beta}\left(\frac{2}{a}\nabla_\alpha u_\beta + \dot{C}^{(t)}_{\alpha\beta}\right), \tag{16}$$

where the gravitational wave part comes from the shear term in equation (2) (Noh & Hwang 2004, 2005; Hwang & Noh 2005a) and it follows equation (14); in order to derive these equations we also used the $\tilde{G}^0_\alpha$-component (momentum constraint) of Einstein's field equations. By combining these equations we have

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta = -\frac{1}{a^2}\frac{\partial}{\partial t}\left[a\nabla \cdot (\delta\boldsymbol{u})\right] + \frac{1}{a^2}\nabla \cdot (\boldsymbol{u} \cdot \nabla\boldsymbol{u}) + \dot{C}^{(t)\alpha\beta}\left(\frac{2}{a}\nabla_\alpha u_\beta + \dot{C}^{(t)}_{\alpha\beta}\right), \tag{17}$$

which also follows from equation (3). Equations (15)–(17) are our extensions of equations (11)–(13) to second-order perturbations in Einstein's theory. We will show that, except for gravitational waves, *exactly the same* equations also follow from Newton's theory. The presence of the gravitational waves, however, can be regarded as one of the truly relativistic effects of gravitation.

In the Newtonian context, the mass and the momentum conservations and Poisson's equation give (Peebles 1980)

$$\dot{\delta} + \frac{1}{a}\nabla \cdot \boldsymbol{u} = -\frac{1}{a}\nabla \cdot (\delta\boldsymbol{u}), \tag{18}$$

$$\dot{\boldsymbol{u}} + \frac{\dot{a}}{a}\boldsymbol{u} + \frac{1}{a}\nabla\delta\Phi = -\frac{1}{a}\boldsymbol{u} \cdot \nabla\boldsymbol{u}, \tag{19}$$

$$\frac{1}{a^2}\nabla^2\delta\Phi = 4\pi G\varrho\delta, \tag{20}$$

where $\delta\Phi$ is the perturbed gravitational potential, $\boldsymbol{u}$ is the perturbed velocity, and $\delta \equiv \delta\varrho/\varrho$. Equation (15) is the same as equation (18); equation (16), ignoring gravitational waves, follows from equations (19) and (20). Thus, equation (17) also naturally follows in Newton's theory (Peebles 1980). This shows the *exact* relativistic–Newtonian correspondence to second order, except for the gravitational wave contribution which is a pure relativistic effect. This also justifies our identifications made in equation (10) to second order. Although we identified the relativistic density and velocity perturbation variables, we *cannot* identify a relativistic variable that corresponds to $\delta\Phi$ to second order (Hwang & Noh 2005a). We believe that this can be understood naturally because Poisson's equation indeed reveals the action-at-a-distance nature and the static nature of Newton's gravitational theory compared with Einstein's (Fock 1964; Rindler 1977). Poisson's equation was formulated in 1812 which was 125 yr after the publication of Newton's *Principia* in 1687. Notice that equations (18)–(20) are valid to *fully non-linear* order. In our relativistic case, however, equations (15)–(17) are valid only to the second order of perturbations.

5 THIRD-ORDER PERTURBATIONS AND PURE RELATIVISTIC CORRECTIONS

Since the zero-pressure Newtonian system is exact to second order in non-linearity, all non-vanishing third- and higher-order perturbation terms in the relativistic analysis can be regarded as the pure general relativistic corrections. Thus we have a clear reason to go to third order, which has not previously been attempted. For simplicity we ignore the gravitational wave contribution; for a complete presentation, see Hwang & Noh (2005b). Based on our success in the second-order perturbations, we continue identifying that equation (10) is valid even to third order, and we will take the consequent additional third-order terms as the pure relativistic corrections. To third order, the perturbed parts of equations (1) and (2) give (Hwang & Noh 2005b)

$$\dot{\delta} + \frac{1}{a}\nabla\cdot\boldsymbol{u} = -\frac{1}{a}\nabla\cdot(\delta\boldsymbol{u}) + \frac{1}{a}[2\varphi\boldsymbol{u} - \nabla(\Delta^{-1}X)]\cdot\nabla\delta, \qquad (21)$$

$$\frac{1}{a}\nabla\cdot\left(\dot{\boldsymbol{u}} + \frac{\dot{a}}{a}\boldsymbol{u}\right) + 4\pi G\mu\delta = -\frac{1}{a^2}\nabla\cdot(\boldsymbol{u}\cdot\nabla\boldsymbol{u}) - \frac{2}{3a^2}\varphi\boldsymbol{u}\cdot\nabla(\nabla\cdot\boldsymbol{u}) + \frac{4}{a^2}\nabla\cdot\left[\varphi\left(\boldsymbol{u}\cdot\nabla\boldsymbol{u} - \frac{1}{3}\boldsymbol{u}\nabla\cdot\boldsymbol{u}\right)\right] - \frac{\Delta}{a^2}[\boldsymbol{u}\cdot\nabla(\Delta^{-1}X)] + \frac{1}{a^2}\boldsymbol{u}\cdot\nabla X + \frac{2}{3a^2}X\nabla\cdot\boldsymbol{u}, \qquad (22)$$

where

$$X \equiv 2\varphi\nabla\cdot\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\varphi + \frac{3}{2}\Delta^{-1}\nabla\cdot[\boldsymbol{u}\cdot\nabla(\nabla\varphi) + \boldsymbol{u}\Delta\varphi]. \qquad (23)$$

In order to derive these equations we also used the $\tilde{G}^0_\alpha$-component of Einstein's field equations. Equations (21) and (22) extend equations (15) and (16) to third order. By combining equations (21) and (22) we can derive

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta = -\frac{1}{a^2}\frac{\partial}{\partial t}[a\nabla\cdot(\delta\boldsymbol{u})] + \frac{1}{a^2}\nabla\cdot(\boldsymbol{u}\cdot\nabla\boldsymbol{u}) + \frac{1}{a^2}\frac{\partial}{\partial t}\{a[2\varphi\boldsymbol{u} - \nabla(\Delta^{-1}X)]\cdot\nabla\delta\} + \frac{2}{3a^2}\varphi\boldsymbol{u}\cdot\nabla(\nabla\cdot\boldsymbol{u}) - \frac{4}{a^2}\nabla\cdot\left[\varphi\left(\boldsymbol{u}\cdot\nabla\boldsymbol{u} - \frac{1}{3}\boldsymbol{u}\nabla\cdot\boldsymbol{u}\right)\right] + \frac{\Delta}{a^2}[\boldsymbol{u}\cdot\nabla(\Delta^{-1}X)] - \frac{1}{a^2}\boldsymbol{u}\cdot\nabla X - \frac{2}{3a^2}X\nabla\cdot\boldsymbol{u}, \qquad (24)$$

which extends equation (17) to third order. The last five lines of equation (24) are pure third-order terms. The variable φ is a perturbed-order metric variable in equation (4) in our comoving gauge condition.

The third-order correction terms in equations (21)–(24) reveal that all of them are simply of φ-order higher than the second-order terms. Thus, the pure general relativistic effects are at least φ-order higher than the relativistic/Newtonian ones in second order. Our φ is related to the perturbed three-space curvature (in our comoving gauge) and is dimensionless (Bardeen 1980). As we mentioned earlier, φ in the comoving gauge is the same as a unique gauge-invariant combination. To linear order such a combination was first introduced by Field & Shepley (1968). For an explicit form of the combination to second order, see equation (281) in Noh & Hwang (2004). Notice that we only need the behaviour of φ to linear order. To linear order, in terms of known Newtonian variables we have (Hwang & Noh 2005b)

$$\varphi = -\delta\Phi + \dot{a}\Delta^{-1}\nabla\cdot\boldsymbol{u}, \qquad (25)$$

and it satisfies (Hwang & Noh 1999a)

$$\dot{\varphi} = 0, \qquad (26)$$

thus $\varphi = C(\boldsymbol{x})$ with *no* decaying mode; this is true considering the presence of the cosmological constant. For $\Lambda = 0$, the temperature anisotropy of cosmic microwave background radiation gives (Sachs & Wolfe 1967; Hwang & Noh 1999b)

$$\frac{\delta T}{T} \sim \frac{1}{3}\delta\Phi \sim \frac{1}{5}\varphi. \qquad (27)$$

The observations of cosmic microwave background radiation give $\delta T/T \sim 10^{-5}$ (Smoot et al. 1992; Spergel et al. 2003), thus

$$\varphi \sim 5\times 10^{-5}, \qquad (28)$$

in the large-scale limit near the horizon scale where $GM/(\lambda c^2) \sim \lambda^2/\lambda_H^2$ approaches unity. Therefore, to third order, the pure relativistic corrections are *independent* of the horizon scale and depend on the linear-order curvature φ ($\sim$gravitational potential $\delta\Phi$) perturbation strength *only*, and are small. That is, compared with the second-order terms, the third-order correction terms in equations (21)–(23) only involve φ, and do not contain terms like $(aH)^{-1}\nabla\varphi$, etc.

6 DISCUSSION

We have shown that to second order, except for the gravitational wave contribution, the zero-pressure general relativistic cosmological perturbation equations can be exactly identified with the known equations in Newton's theory. As a consequence, to second order, we identified the correct relativistic variables which can be interpreted as density $\delta\mu$ and velocity $\delta\theta$ perturbations in equation (10). We also showed that to second order, the Newtonian hydrodynamic equations remain valid on all cosmological scales including the super-horizon scale. More precisely, the relativistic equations can be identified with the continuity equation and with the divergence of the momentum conservation equation replacing the Newtonian gravitational potential using Poisson's equation. It might also happen that our relativistic results give relativistic correction terms appearing to second order, as we approach and go beyond the horizon scale, that are strongly relativistic regimes. Our results show that there are no such correction terms appearing to second order, and except for gravitational waves, the correspondence is exact to that order. Ignoring gravitational waves, the pure relativistic correction terms start appearing from third order. Our study shows that, to third order, the correction terms only involve φ which is again independent of the horizon scale and is small on large scales.

In the non-linear clustered regions, we may have $\varphi \sim \delta\Phi \sim GM/(Rc^2)$ where M and R are the characteristic mass and length scales involved. In such clustered regimes the post-Newtonian approximation would complement our non-linear perturbation approach. The post-Newtonian approximation takes v/c-expansion when the motions are slow, $(v/c)^2 \ll 1$, and gravity is weak, $GM/(Rc^2) \ll 1$. Thus, this approximation is valid in the fully non-linear case under a weakly relativistic situation, which can be compared with the relativistic non-linear perturbation theory; the latter is valid in the fully relativistic case under the weakly non-linear situation. A complementary result, showing the relativistic–Newtonian correspondence in the Newtonian limit of the post-Newtonian approach, can be found in Kofman & Pogosyan (1995) (see also Bertschinger & Hamilton 1994; Ellis & Dunsby 1997). In fact, the Newtonian hydrodynamic equations naturally appear in the zeroth-order post-Newtonian approximation (Chandrasekhar 1965). Recently, we presented the fully non-linear cosmological hydrodynamic equations with first-order post-Newtonian correction terms (Hwang, Noh & Puetzfeld 2005); we showed that these correction terms have typically $GM/(Rc^2) \sim (v/c)^2 \sim 10^{-5}$ times smaller than the Newtonian terms in the non-linearly clustered regions.

Therefore, our general relativistic results allow us to draw the following important practical conclusion which is stated in our title. As we prove that the Newtonian hydrodynamic equations are valid on all cosmological scales to second order, and that the third-order pure relativistic correction terms are small and independent of the horizon, one can now use the large-scale Newtonian numerical simulation more reliably as the simulation scale approaches and even goes beyond the horizon. The fluctuations near the horizon scale are supposed to be linear or weakly non-linear; otherwise, it is difficult to introduce the spatially homogeneous and isotropic background world model which is the basic assumption of modern cosmology. In the small-scale but fully non-linear stage, the post-Newtonian approximation also shows that the relativistic correction terms are small, thus the Newtonian simulations can be trusted. The sub-horizon-scale Newtonian non-linear inhomogeneities are not supposed to affect the homogeneous and isotropic background world model (Siegel & Fry 2005). The other side of this conclusion is that it might be difficult to find testable signatures of Einstein's gravity theory based on such large-scale weakly non-linear structures (with relativistic corrections) or small-scale fully non-linear structures (with post-Newtonian corrections). However, it would be interesting to find cosmological situations in which the pure relativistic correction terms in equations (21)–(24) or the first-order post-Newtonian corrections terms derived in Hwang et al. (2005) could have observationally distinguishable consequences. Since our equations include the cosmological constant, our equations and conclusions are relevant to the currently favoured world models.

In our relativistic–Newtonian correspondence to second order, the relativistic equations are identified with the continuity equation and the divergence of the Euler equation replacing the Newtonian gravitational potential using Poisson's equation. It is important to remember that we showed the relativistic–Newtonian correspondence for the density and velocity perturbations, but not for the gravitational potential. Therefore, although our result shows that one can trust cold dark matter simulations at all scales for the density and velocity fields, it does *not* imply that one can trust the Newtonian simulations for effects involving the gravitational potential, like the weak gravitational lensing effects. In order to handle lensing effects properly we often require an extra factor of 2 which, indeed, comes from the post-Newtonian effects. According to White, the Newtonian simulations are normally employed for lensing studies by tracing rays through the simulation (discretized into a set of lens planes along the observer's past light-cone) using the lowest post-Newtonian approximation for the deflection on each lens plane; this could include the standard factor of 2 in the light deflection formula (White, private communication).

Our relativistic-Newtonian correspondence to the second-order perturbation is valid for the scalar-type perturbation assuming a single-component, zero-pressure and irrotational fluid in the flat cosmological background. Dropping any of these conditions could potentially lead to relativistic corrections. The genuine relativistic correction terms appear as we consider the gravitational waves to second order. We showed that pure relativistic correction terms in the scalar-type perturbation appear at third order; we showed that these correction terms do not involve the horizon scale and are small in our observable patch of the Universe. It will be interesting to see the effects of the radiation pressure and the time-varying dark energy (modelled by a minimally coupled scalar field) to second order in perturbations. It is known that the effects of general relativistic pressure cannot be simulated by Newtonian treatment even to linear order (Sachs & Wolfe 1967); this is also true for the scalar field. Thus, we anticipate that pure general relativistic effects will be present to second order from both the pressure and the time-varying dark energy. Extensions to include the pressure, the rotation, the non-flat background and the multi-component situation will be investigated on future occasions.

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REFERENCES

Abazajian K. et al., 2004, http://www.sdss.org/dr3
Bardeen J. M., 1980, Phys. Rev. D, 22, 1882
Bardeen J. M., 1988, in Fang L., Zee A., eds, Particle Physics and Cosmology. Gordon and Breach, London, p. 1
Bernardeau F., Colombi S., Gaztanaga E., Scoccimarro R., 2002, Phys. Rep., 367, 1
Bertschinger E., 1998, ARA&A, 36, 599
Bertschinger E., Hamilton A. J. S., 1994, ApJ, 435, 1
Bode P., Ostriker J. P., 2003, ApJS, 145, 1
Bonnor W. B., 1957, MNRAS, 117, 104
Chandrasekhar S., 1965, ApJ, 142, 1488
Colberg J. M. et al., 2000, MNRAS, 319, 209
Colless M. et al., 2003, http://www.mso.anu.edu.au/2dFGRS
Cooray A., Sheth R., 2002, Phys. Rep., 372, 1
Dubinski J., Kim J., Park C., Humble R., 2003, New Astron., 9, 111
Ehlers J., 1961, Proc. Mainz Acad. Sci. Lit., No. 11, 792 (translated in 1993, Gen. Relativ. Gravitation, 25, 1225)
Einstein A., 1917, Sitzungsb. Preuss. Akad. Wiss., 142 (translated in Bernstein J., Feinberg G., eds, 1986, Cosmological Constants: Papers in Modern Cosmology. Columbia Univ. Press, New York, p. 16)
Ellis G. F. R., 1971, in Sachs R. K., ed., General Relativity and Cosmology. Academic Press, New York, p. 104
Ellis G. F. R., 1973, in Schatzmann E., ed., Cargese Lectures in Physics. Gorden and Breach, New York, p. 1
Ellis G. F. R., Dunsby P. K. S., 1997, ApJ, 479, 97

Evrard A. E. et al., 2002, ApJ, 573, 7
Field G. B., Shepley L. C., 1968, Ap&SS, 1, 309
Fock V., 1964, The Theory of Space, Time, and Gravitation, 2nd edn. Macmillan, New York
Friedmann A. A., 1922, Z. Phys., 10, 377 (translated in Bernstein J., Feinberg G., eds, 1986, Cosmological Constants: Papers in Modern Cosmology. Columbia Univ. Press, New York, p. 49)
Friedmann A. A., 1924, Z. Phys., 21, 326 (translated in Bernstein J., Feinberg G., eds. 1986, Cosmological constants: papers in modern cosmology. Columbia Univ. Press, New York, p. 59)
Hwang J., 1991, ApJ, 375, 443
Hwang J., Noh H., 1999a, Gen. Relativ. Gravitation, 31, 1131
Hwang J., Noh H., 1999b, Phys. Rev. D, 59, 067302
Hwang J., Noh H., 2005a, Phys. Rev. D, 72, 044011
Hwang J., Noh H., 2005b, Phys. Rev. D, 72, 044012
Hwang J., Noh H., Puetzfeld D., 2005, Phys. Rev. D submitted (astro-ph/0507085)
Jenkins A., Frenk C. S., White S. D. M., Colberg J. M., Cole S., Evrard A. E., Couchman H. M. P., Yoshida N., 2001, MNRAS, 321, 372
Kofman L., Pogosyan D., 1995, ApJ, 442, 30
Layzer D., 1954, AJ, 59, 268
Lemons D. S., 1988, Am. J. Phys., 56, 502
Lifshitz E. M., 1946, J. Phys. (USSR), 10, 116
McCrea W. H., Milne E. A., 1934, Quart. J. Math., 5, 73
Milne E. A., 1934, Quart. J. Math., 5, 64
Noh H., Hwang J., 2004, Phys. Rev. D, 69, 104011
Noh H., Hwang J., 2005, Class. Quantum. Grav., 22, 3181
Park C., Kim J., Gott J. R., III, 2005, ApJ, 633, 1
Peebles P. J. E., 1980, The Large-Scale Structure of the Universe. Princeton Univ. Press, Princeton, NJ
Rindler W., 1977, Essential Relativity. Springer-Verlag, New York
Robertson H. P., 1929, Proc. Natl Acad. Sci., 15, 822
Sachs R. K., Wolfe A. M., 1967, ApJ, 147, 73
Sahni V., Coles P., 1995, Phys. Rep., 262, 1
Siegel E. R., Fry J. N., 2005, ApJ, 628, L1
Smoot G. F. et al., 1992, ApJ, 396, L1
Spergel D. N. et al., 2003, ApJS, 148, 175
Tegmark M. et al., 2004, Phys. Rev. D, 69, 103501

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