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Second-order perturbations of cosmological fluids: Relativistic effects of pressure, multicomponent, curvature, and rotation

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We present general relativistic correction terms appearing in Newton's gravity to the second-order perturbations of cosmological fluids. In our previous work we have shown that to the second-order perturbations, the density and velocity perturbation equations of general relativistic zero-pressure, irrotational, single-component fluid in a spatially flat background *coincide exactly* with the ones known in Newton's theory without using the gravitational potential. We also have shown the effect of gravitational waves to the second order, and pure general relativistic correction terms appearing in the third-order perturbations. Here, we present results of second-order perturbations relaxing all the assumptions made in our previous works. We derive the general relativistic correction terms arising due to (i) pressure, (ii) multicomponent, (iii) background spatial curvature, and (iv) rotation. In the case of multicomponent zero-pressure, irrotational fluids under the flat background, we effectively do not have relativistic correction terms, thus the relativistic equations expressed in terms of density and velocity perturbations again *coincide* with the Newtonian ones. In the other three cases we generally have pure general relativistic correction terms. In the case of pressure, the relativistic corrections appear even in the level of background and linear perturbation equations. In the presence of background spatial curvature, or rotation, pure relativistic correction terms directly appear in the Newtonian equations of motion of density and velocity perturbations to the second order; to the linear order, without using the gravitational potential (or metric perturbations), we have relativistic/Newtonian correspondences for density and velocity perturbations of a single-component fluid including the rotation even in the presence of background spatial curvature. In the small-scale limit (far inside the horizon), to the second-order, relativistic equations of density and velocity perturbations including the rotation *coincide* with the ones in Newton's gravity. All equations in this work include the cosmological constant in the background world model. We emphasize that our relativistic/Newtonian correspondences in several situations and pure general relativistic corrections in the context of Newtonian equations are mainly about the dynamic equations of density and velocity perturbations *without* using the gravitational potential (metric perturbations). Consequently, our relativistic/Newtonian correspondences do *not* imply the absence of many space-time (i.e., pure general relativistic) effects like frame dragging, and redshift and deflection of photons even in such cases. We also present the case of multiple minimally coupled scalar fields, and properly derive the large-scale conservation properties of curvature perturbation variable in various temporal gauge conditions to the second order.

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I. INTRODUCTION

Large amount of cosmological data on the large-scale structures and motions of galaxies [1,2], and the temperature and polarization anisotropies of cosmic microwave background radiation [3,4] have been accumulating recently. In the current standard cosmological scenario such structures are explained as small (linear) or large (nonlinear) deviations from a spatially homogeneous and isotropic Friedmann background world model. In order to explain these data theoretically, researchers rely on the linear perturbation theory based on relativistic gravity, and quasilinear perturbation theories and nonlinear simulations based on Newton's gravity. To the linear order in perturbation the general relativistic result was first derived by Lifshitz in 1946 [5], and later shown to coincide with the Newtonian result in a zero-pressure medium [6]. The same is also known to be true for the background world model. That is, the general relativistic result was first derived by Friedmann in 1922 [7], and later shown to coincide with the Newtonian result in a zero-pressure medium [8]. Compared with the relativistic cosmological linear perturbation theory which is now well developed [9–15], the general relativistic nonlinear or quasilinear perturbation theories have not been well developed in the literature. Consequently, it has not been clear whether we can rely on Newtonian theory in handling the large-scale cosmic structures. As more precise observational data are rapidly accumulating, it is likely that soon we need quantitative estimations of the potential general relativistic corrections to the Newtonian treatment. Our aim in this

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work is to provide such general relativistic corrections appearing in the second-order perturbations.

The observed large-scale distribution of galaxies shows that in the largest observed scale (say, larger than several hundred megaparsec scale) the distribution may not be inconsistent with the linear assumption around the Friedmann background. However, as the scale becomes smaller the distribution apparently shows quasilinear to fully nonlinear structures. The fully nonlinear processes occur in small scale where the relativistic effects characterized by $GM/Rc^2 \sim v^2/c^2$ are quite small. If we could ignore such relativistic effects, Newton's gravity would be sufficient to handle the relevant nonlinear processes. If we need to consider the weakly relativistic correction terms in a fully nonlinear stage, instead of the relativistic perturbation approach which can handle the fully relativistic processes under weakly nonlinear assumption, we can use the post-Newtonian approximation developed in the context of cosmology in [16].

For structures in the quasilinear evolution phase, previous researches were based on Newton's gravity especially assuming the single-component zero-pressure fluid without rotational perturbation in a flat background [17]. In our previous works in [18,19] we have shown that, in the single-component zero-pressure fluid without rotational perturbation, cosmological scalar-type perturbation equations in a spatially flat background coincide exactly with the Newtonian ones up to the second order in perturbation. In [18,19] we also have shown the contribution of gravitational wave perturbations to the hydrodynamic parts in the second-order perturbations. In Newton's gravity the hydrodynamic equations of zero-pressure fluid contain only the quadratic order nonlinearity. In [20] we presented pure general relativistic correction terms appearing in the third-order perturbation, and showed that all third-order correction terms are 10^{-5} times smaller than the second-order relativistic/Newtonian terms, and independent of the horizon scale.

In this work we will take into account the pure general relativistic effects appearing in the second-order perturbations which were ignored in our previous work in [19]. We will consider general relativistic effects of (i) pressure, (ii) multicomponent, (iii) background curvature, (iv) rotation in cosmological fluids to the second-order perturbations. As results we will show that, although in [19] we have shown the exact relativistic/Newtonian correspondence to the second-order perturbations by ignoring the above four conditions and the gravitational waves, as we take these four effects into account we often encounter pure general relativistic effects appearing in the corresponding Newtonian equations even to the second order in perturbations. Our results will show that the relativistic/Newtonian correspondence continues even in the multicomponent situation assuming zero-pressure irrotational fluid in a flat background, but in the presence of the cosmological constant. This is a practically useful result because the matter content of the present universe is dominated by the collisionless cold dark matter and the baryon both of which practically have zero pressure. In the other three cases, relaxing any of the assumptions about pressure, rotation, and background curvature generally leads to pure general relativistic correction terms to the second order. We will present such correction terms in the context of Newtonian hydrodynamics. One additional relativistic/Newtonian correspondence occurs in the case of rotation in small-scale (sub-horizon-scale) limit which is another practically important result. This correspondence allows us to use the Newtonian equations safely in such a small-scale limit even in the presence of rotational perturbation to the second order.

In Sec. II we summarize Newtonian hydrodynamic perturbation equations valid to fully nonlinear order. In Sec. III we briefly summarize our previous result of relativistic/Newtonian correspondence to the second order, and pure general relativistic correction terms appearing in the third order. In Sec. IV we present parts of the covariant and the Arnowitt-Deser-Misner (ADM) equations which are valid in a multicomponent situation. In Sec. V we present the basic perturbation equations valid to second order. In [18] the basic set of equations was presented using fluid quantities based on the normal-frame four-vector. The fluid quantities in the present work are based on the energy-frame four-vector, and in this section we present the basic equations using such fluid quantities. In Secs. VI, VII, VIII, and IX we analyze the effects of the pressure, the multicomponents, the background curvature, and the rotational perturbation, respectively. In Sec. X we present equations in the scalar fields and generalized gravity theories using the energy-frame fluid quantities. In Sec. XI we properly derive conservation properties of curvature perturbation in various temporal gauge (hypersurface) conditions to the second order in perturbations. Section XII is a discussion. In this work we follow notations used in [18,19]. We set $c \equiv 1$, but when we compare with the Newtonian case we often recover the speed of light c.

II. NEWTONIAN NONLINEAR PERTURBATIONS

In order to compare properly the relativistic results with the Newtonian ones, in this section we summarize the Newtonian cosmological perturbation theory in fully nonlinear context. We consider multicomponent fluids in the presence of isotropic pressure. In the case of n fluids with the mass densities ϱ_i, the pressures p_i, the velocities $\mathbf{v}_i$ ($i = 1, 2, \dots, n$), and the gravitational potential Φ, we have

$$\dot{\varrho}_i + \nabla \cdot (\varrho_i \mathbf{v}_i) = 0, \tag{1}$$

$$\dot{\mathbf{v}}_i + \mathbf{v}_i \cdot \nabla \mathbf{v}_i = -\frac{1}{\varrho_i} \nabla p_i - \nabla \Phi, \tag{2}$$

$$\nabla^2\Phi = 4\pi G \sum_{j=1}^{n} \varrho_j. \tag{3}$$

Assuming the presence of a spatially homogeneous and isotropic but temporally dynamic background, we introduce fully nonlinear perturbations as

$$\begin{aligned} \varrho_i &= \bar{\varrho}_i + \delta\varrho_i, \qquad p_i = \bar{p}_i + \delta p_i, \\ \mathbf{v}_i &= H\mathbf{r} + \mathbf{u}_i, \qquad \Phi = \bar{\Phi} + \delta\Phi, \end{aligned} \tag{4}$$

where $H \equiv \dot{a}/a$, and $a(t)$ is a cosmic scale factor. We move to the comoving coordinate $\mathbf{x}$ where

$$\mathbf{r} \equiv a(t)\mathbf{x}, \tag{5}$$

thus

$$\begin{aligned} &\nabla = \nabla_{\mathbf{r}} = \frac{1}{a}\nabla_{\mathbf{x}}, \\ &\frac{\partial}{\partial t} = \frac{\partial}{\partial t}\Big|_{\mathbf{r}} = \frac{\partial}{\partial t}\Big|_{\mathbf{x}} + \left(\frac{\partial}{\partial t}\Big|_{\mathbf{r}}\mathbf{x}\right)\cdot\nabla_{\mathbf{x}} = \frac{\partial}{\partial t}\Big|_{\mathbf{x}} - H\mathbf{x}\cdot\nabla_{\mathbf{x}}. \end{aligned} \tag{6}$$

In the following we neglect the subindex $\mathbf{x}$. To the background order Eqs. (1)–(3) give

$$\begin{aligned} &\dot{\varrho}_i + 3H\varrho_i = 0, \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\sum_j \varrho_j, \\ &H^2 = \frac{8\pi G}{3}\sum_j \varrho_j + \frac{2E}{a^2}, \end{aligned} \tag{7}$$

where E is an integration constant which can be interpreted as the specific total energy in Newton's gravity; in Einstein's gravity we have $2E = -Kc^2$ where K can be normalized, using a, to become the sign of spatial curvature. To the perturbed order Eqs. (1)–(3), respectively, give [13]

$$\dot{\delta}_i + \frac{1}{a}\nabla\cdot\mathbf{u}_i = -\frac{1}{a}\nabla\cdot(\delta_i\mathbf{u}_i), \tag{8}$$

$$\dot{\mathbf{u}}_i + H\mathbf{u}_i + \frac{1}{a}\mathbf{u}_i\cdot\nabla\mathbf{u}_i = -\frac{1}{a\bar{\varrho}_i}\frac{\nabla\delta p_i}{1+\delta_i} - \frac{1}{a}\nabla\delta\Phi, \tag{9}$$

$$\frac{1}{a^2}\nabla^2\delta\Phi = 4\pi G\sum_j \bar{\varrho}_j\delta_j. \tag{10}$$

By introducing the expansion θ_i and the rotation $\vec{\omega}_i$ of each component as

$$\theta_i \equiv \frac{1}{a}\nabla\cdot\mathbf{u}_i, \qquad \vec{\omega}_i \equiv \frac{1}{a}\nabla\times\mathbf{u}_i, \tag{11}$$

Eq. (9) gives

$$\dot{\theta}_i + 2H\theta_i + 4\pi G\sum_j \bar{\varrho}_j\delta_j = -\frac{1}{a^2}\nabla\cdot(\mathbf{u}_i\cdot\nabla\mathbf{u}_i) - \frac{1}{a^2\bar{\varrho}_i}\nabla\cdot\left(\frac{\nabla\delta p_i}{1+\delta_i}\right), \tag{12}$$

$$\dot{\vec{\omega}}_i + 2H\vec{\omega}_i = -\frac{1}{a^2}\nabla\times(\mathbf{u}_i\cdot\nabla\mathbf{u}_i) + \frac{1}{a^2\bar{\varrho}_i}\times\frac{(\nabla\delta_i)\times\nabla\delta p_i}{(1+\delta_i)^2}. \tag{13}$$

By introducing decomposition of perturbed velocity into the potential and transverse parts as

$$\begin{aligned} &\mathbf{u}_i \equiv \nabla u_i + \mathbf{u}_i^{(v)}, \qquad \nabla\cdot\mathbf{u}_i^{(v)} \equiv 0; \\ &\theta_i = \frac{\Delta}{a}u_i, \qquad \vec{\omega}_i = \frac{1}{a}\nabla\times\mathbf{u}_i^{(v)}, \end{aligned} \tag{14}$$

instead of Eq. (13) we have

$$\dot{\mathbf{u}}_i^{(v)} + H\mathbf{u}_i^{(v)} = -\frac{1}{a}\left[\mathbf{u}_i\cdot\nabla\mathbf{u}_i + \frac{1}{\bar{\varrho}_i}\frac{\nabla\delta p_i}{1+\delta_i} - \nabla\Delta^{-1}\nabla\cdot\left(\mathbf{u}_i\cdot\nabla\mathbf{u}_i + \frac{1}{\bar{\varrho}_i}\frac{\nabla\delta p_i}{1+\delta_i}\right)\right]. \tag{15}$$

We note that the pure u_i contributions in the right-hand side of Eq. (13) or Eq. (15) vanish. Thus, under vanishing pressure, pure irrotational perturbation *cannot* generate the rotational perturbation. Equation (13) shows that presence of pressure perturbation oblique (i.e., nonparallel) to the density perturbation can generate rotational perturbation. Combining Eqs. (8) and (12) we can derive

$$\ddot{\delta}_i + 2H\dot{\delta}_i - 4\pi G\sum_j \bar{\varrho}_j\delta_j = -\frac{1}{a^2}[a\nabla\cdot(\delta_i\mathbf{u}_i)]^{\cdot} + \frac{1}{a^2}\nabla\cdot(\mathbf{u}_i\cdot\nabla\mathbf{u}_i) + \frac{1}{a^2\bar{\varrho}_i}\nabla\cdot\left(\frac{\nabla\delta p_i}{1+\delta_i}\right). \tag{16}$$

Equations (8)–(16) are valid to fully nonlinear order. Notice that for vanishing pressure these equations have only quadratic order nonlinearity in perturbations.

III. SUMMARY OF PREVIOUS WORK

In [18,19] we have derived second-order perturbation equations in Einstein's gravity valid for the single-component, irrotational, and zero-pressure medium in zero-curvature background. These are

$$\dot{\delta} + \frac{1}{a}\nabla\cdot\mathbf{u} = -\frac{1}{a}\nabla\cdot(\delta\mathbf{u}), \tag{17}$$

$$\frac{1}{a}\nabla\cdot(\dot{\mathbf{u}}+H\mathbf{u})+4\pi G\varrho\delta=-\frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u})-\dot{C}^{(t)\alpha\beta}\Big(\frac{2}{a}u_{\alpha|\beta}+\dot{C}^{(t)}_{\alpha\beta}\Big), \tag{18}$$

$$\ddot{\delta}+2H\dot{\delta}-4\pi G\varrho\delta=\frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u})-\frac{1}{a^2}[a\nabla\cdot(\delta\mathbf{u})]^{\cdot}+\dot{C}^{(t)\alpha\beta}\Big(\frac{2}{a}u_{\alpha|\beta}+\dot{C}^{(t)}_{\alpha\beta}\Big). \tag{19}$$

Except for the presence of tensor-type perturbation, Eqs. (17)–(19) are exactly the same as the ones known in the Newtonian theory; compare with Eqs. (8), (12), and (16) without the i indices. We note that these equations are valid in the presence of Λ. To the linear order, these are valid in the presence of general K, see Sec. VIII. In order to derive these relativistic equations we have correctly identified the relativistic density and velocity perturbation variables which correspond to the Newtonian counterparts to the second order. In the relativistic context, our δ and $\mathbf{u}$ are the density perturbation and velocity perturbation (related to) the perturbed expansion scalar, respectively, in the comoving gauge; the variables are equivalently gauge invariant. However, we were not able to identify a relativistic variable which corresponds to the Newtonian gravitational potential to the second order; this is understandable if we consider the factor of 2 difference between Einstein's (post-Newtonian) and Newton's gravity theories in predicting the light bending under the gravitational field (potential). Only to the linear order the "spatial curvature perturbation in the zero-shear gauge" ($\varphi_\chi\equiv\varphi-H\chi$, a gauge-invariant combination which becomes φ in the zero-shear gauge $\chi\equiv 0$, see later) can be identified as the perturbed Newtonian potential [9,12]. Equations (17)–(19) include effects of gravitational waves to the density and velocity perturbations. Equations of the gravitational waves can be found in [19].

To the third order, we have [20]

$$\dot{\delta}+\frac{1}{a}\nabla\cdot\mathbf{u}=-\frac{1}{a}\nabla\cdot(\delta\mathbf{u})+\frac{1}{a}[2\varphi_v\mathbf{u}-\nabla(\Delta^{-1}X)]\cdot\nabla\delta, \tag{20}$$

$$\frac{1}{a}\nabla\cdot\Big(\dot{\mathbf{u}}+\frac{\dot{a}}{a}\mathbf{u}\Big)+4\pi G\mu\delta=-\frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u})-\frac{2}{3a^2}\varphi_v\mathbf{u}\cdot\nabla(\nabla\cdot\mathbf{u})+\frac{4}{a^2}\nabla\cdot\Big[\varphi_v\Big(\mathbf{u}\cdot\nabla\mathbf{u}-\frac{1}{3}\mathbf{u}\nabla\cdot\mathbf{u}\Big)\Big]-\frac{\Delta}{a^2}\Big[\mathbf{u}\cdot\nabla(\Delta^{-1}X)\Big]+\frac{1}{a^2}\mathbf{u}\cdot\nabla X+\frac{2}{3a^2}X\nabla\cdot\mathbf{u}, \tag{21}$$

$$\ddot{\delta}+2\frac{\dot{a}}{a}\dot{\delta}-4\pi G\mu\delta=-\frac{1}{a^2}\frac{\partial}{\partial t}[a\nabla\cdot(\delta\mathbf{u})]+\frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u})+\frac{1}{a^2}\frac{\partial}{\partial t}\{a[2\varphi_v\mathbf{u}-\nabla(\Delta^{-1}X)]\cdot\nabla\delta\}+\frac{2}{3a^2}\varphi_v\mathbf{u}\cdot\nabla(\nabla\cdot\mathbf{u})-\frac{4}{a^2}\nabla\cdot\Big[\varphi_v\Big(\mathbf{u}\cdot\nabla\mathbf{u}-\frac{1}{3}\mathbf{u}\nabla\cdot\mathbf{u}\Big)\Big]+\frac{\Delta}{a^2}[\mathbf{u}\cdot\nabla(\Delta^{-1}X)]-\frac{1}{a^2}\mathbf{u}\cdot\nabla X-\frac{2}{3a^2}X\nabla\cdot\mathbf{u}, \tag{22}$$

where

$$X\equiv 2\varphi_v\nabla\cdot\mathbf{u}-\mathbf{u}\cdot\nabla\varphi_v+\tfrac{3}{2}\Delta^{-1}\nabla\cdot[\mathbf{u}\cdot\nabla(\nabla\varphi_v)+\mathbf{u}\Delta\varphi_v]. \tag{23}$$

In these equations we ignored the role of tensor-type perturbation; for a complete set of equations, see [20]. The variable φ_v is a "perturbed-order spatial curvature variable in the comoving gauge condition" (to the linear order, $\varphi_v\equiv\varphi-aHv$, a gauge-invariant combination which becomes φ in the comoving gauge $v\equiv 0$, see later).

All the third-order correction terms in Eqs. (20)–(22) are simply of φ_v-order higher than the second-order relativistic/Newtonian terms. Thus, the pure general relativistic effects are at least φ_v-order higher than the relativistic/Newtonian ones in the second-order equations. Thus, we only need the behavior of φ_v to the linear order which is related to the other hydrodynamic variables as

$$\varphi_v=-\delta\Phi+\dot{a}\Delta^{-1}\nabla\cdot\mathbf{u}. \tag{24}$$

It also satisfies [21]

$$\dot{\varphi}_v=0, \tag{25}$$

thus $\varphi_v=C(\mathbf{x})$ with *no* decaying mode; this is true considering the presence of the cosmological constant, see [21].

IV. RELATIVISTIC FULLY NONLINEAR EQUATIONS

In this section, for convenience, we present some additional covariant or ADM equations not available in [18]. In the multicomponent situation, we have

$$\tilde{T}_{ab}\equiv\sum_j\tilde{T}_{(j)ab}. \tag{26}$$

The energy-momentum conservation gives

$$\tilde{T}_{(i)a;b}^{\ \ \ \ \ b} = \tilde{I}_{(i)a}, \qquad \sum_j \tilde{I}_{(j)a} = 0. \tag{27}$$

Tildes indicate the covariant quantities.

A. Covariant equations

We introduce the fluid quantities as

$$\tilde{T}_{(i)ab} \equiv \tilde{\mu}_{(i)}\tilde{u}_{(i)a}\tilde{u}_{(i)b} + \tilde{p}_{(i)}(\tilde{g}_{ab} + \tilde{u}_{(i)a}\tilde{u}_{(i)b}) + \tilde{q}_{(i)a}\tilde{u}_{(i)b} + \tilde{q}_{(i)b}\tilde{u}_{(i)a} + \tilde{\pi}_{(i)ab}, \tag{28}$$

where

$$\tilde{u}_{(i)}^{\ a}\tilde{u}_{(i)a} \equiv -1, \quad \tilde{u}^a_{(i)}\tilde{q}_{(i)a} \equiv 0 \equiv \tilde{u}^b_{(i)}\tilde{\pi}_{(i)ab}, \quad \tilde{\pi}_{(i)a}^{\ \ \ \ a} \equiv 0. \tag{29}$$

The fluid quantities of each component are based on the fluid four-vector $\tilde{u}_{(i)a}$ as

$$\tilde{\mu}_{(i)} \equiv \tilde{T}_{(i)ab}\tilde{u}^a_{(i)}\tilde{u}^b_{(i)}, \qquad \tilde{p}_{(i)} \equiv \tfrac{1}{3}\tilde{T}_{(i)ab}\tilde{h}^{ab}_{(i)},$$
$$\tilde{q}_{(i)a} \equiv -\tilde{T}_{(i)cd}\tilde{u}^c_{(i)}\tilde{h}_{(i)a}^{\ \ \ \ d}, \tag{30}$$
$$\tilde{\pi}_{(i)ab} \equiv \tilde{T}_{(i)cd}\tilde{h}_{(i)a}^{\ \ \ \ c}\tilde{h}_{(i)b}^{\ \ \ \ d} - \tilde{p}_{(i)}\tilde{h}_{(i)ab},$$

where $\tilde{h}_{(i)ab} \equiv \tilde{g}_{ab} + \tilde{u}_{(i)a}\tilde{u}_{(i)b}$. Equation (27) gives

$$\dot{\tilde{\tilde{\mu}}}_{(i)} + (\tilde{\mu}_{(i)} + \tilde{p}_{(i)})\tilde{\theta}_{(i)} + \tilde{q}^a_{(i);a} + \tilde{q}^a_{(i)}\tilde{a}_{(i)a} + \tilde{\pi}^{ab}_{(i)}\tilde{\sigma}_{(i)ab} = -\tilde{u}^a_{(i)}\tilde{I}_{(i)a}, \tag{31}$$

$$(\tilde{\mu}_{(i)} + \tilde{p}_{(i)})\tilde{a}_{(i)a} + \tilde{h}_{(i)a}^{\ \ \ \ b}(\tilde{p}_{(i),b} + \dot{\tilde{\tilde{q}}}_{(i)b} + \tilde{\pi}_{(i)b;c}^{\ \ \ \ \ \ c}) + \tilde{q}^b_{(i)}(\tilde{\omega}_{(i)ab} + \tilde{\sigma}_{(i)ab} + \tfrac{4}{3}\tilde{\theta}_{(i)}\tilde{h}_{(i)ab}) = \tilde{h}_{(i)a}^{\ \ \ \ b}\tilde{I}_{(i)b}, \tag{32}$$

where the kinematic quantities are also based on the fluid four-vector $\tilde{u}_{(i)a}$ as

$$\tilde{h}_{(i)a}^{\ \ \ \ c}\tilde{h}_{(i)b}^{\ \ \ \ d}\tilde{u}_{(i)c;d} = \tilde{h}_{(i)[a}^{\ \ \ \ \ c}\tilde{h}_{(i)b]}^{\ \ \ \ \ d}\tilde{u}_{(i)c;d} + \tilde{h}_{(i)(a}^{\ \ \ \ \ c}\tilde{h}_{(i)b)}^{\ \ \ \ \ d}\tilde{u}_{(i)c;d}$$
$$\equiv \tilde{\omega}_{(i)ab} + \tilde{\theta}_{(i)ab} = \tilde{u}_{(i)a;b} + \tilde{a}_{(i)a}\tilde{u}_{(i)b},$$
$$\tilde{\theta}_{(i)} \equiv u^a_{(i);a}, \quad \tilde{\sigma}_{(i)ab} \equiv \tilde{\theta}_{(i)ab} - \tfrac{1}{3}\tilde{\theta}_{(i)}\tilde{h}_{(i)ab},$$
$$\tilde{a}_{(i)a} \equiv \tilde{u}_{(i)a;b}\tilde{u}^b_{(i)} \equiv \dot{\tilde{\tilde{u}}}_{(i)a}, \quad \dot{\tilde{\tilde{\mu}}}_{(i)} \equiv \tilde{\mu}_{(i),a}\tilde{u}^a_{(i)}. \tag{33}$$

In the multicomponent situation, we can derive the corresponding equation of Raychaudhury equation for the individual component. From $\tilde{u}_{(i)a;bc} - \tilde{u}_{(i)a;cb} \equiv \tilde{u}_{(i)d}\tilde{R}^d_{\ abc}$ we can derive

$$\dot{\tilde{\tilde{\theta}}}_{(i)} + \tfrac{1}{3}\tilde{\theta}^2_{(i)} - \tilde{a}^a_{(i);a} + \tilde{\sigma}^{ab}_{(i)}\tilde{\sigma}_{(i)ab} - \tilde{\omega}^{ab}_{(i)}\tilde{\omega}_{(i)ab} = 4\pi G(\tilde{\mu} - 3\tilde{p} - 2\tilde{T}_{ab}\tilde{u}^a_{(i)}\tilde{u}^b_{(i)}) + \Lambda. \tag{34}$$

In the energy frame we take $\tilde{q}_{(i)a} \equiv 0$ for each component of the fluids without losing any physical degree of freedom.

In a single-component situation, taking the energy frame, the energy conservation equation, the momentum conservation equation, and the Raychaudhury equation are [22]

$$\dot{\tilde{\tilde{\mu}}} + (\tilde{\mu} + \tilde{p})\tilde{\theta} + \tilde{\pi}^{ab}\tilde{\sigma}_{ab} = 0, \tag{35}$$

$$(\tilde{\mu} + \tilde{p})\tilde{a}_a + \tilde{h}^b_a(\tilde{p}_{,b} + \tilde{\pi}^c_{b;c}) = 0, \tag{36}$$

$$\dot{\tilde{\tilde{\theta}}} + \tfrac{1}{3}\tilde{\theta}^2 - \tilde{a}^a_{\ ;a} + \tilde{\sigma}^{ab}\tilde{\sigma}_{ab} - \tilde{\omega}^{ab}\tilde{\omega}_{ab} + 4\pi G(\tilde{\mu} + 3\tilde{p}) - \Lambda = 0. \tag{37}$$

By combining Eqs. (35)–(37) we can derive

$$\left(\frac{\dot{\tilde{\tilde{\mu}}} + \tilde{\pi}^{ab}\tilde{\sigma}_{ab}}{\tilde{\mu} + \tilde{p}}\right)^{\tilde{\cdot}} - \frac{1}{3}\left(\frac{\dot{\tilde{\tilde{\mu}}} + \tilde{\pi}^{ab}\tilde{\sigma}_{ab}}{\tilde{\mu} + \tilde{p}}\right)^2 = 4\pi G(\tilde{\mu} + 3\tilde{p}) - \Lambda + \tilde{\sigma}^{ab}\tilde{\sigma}_{ab} - \tilde{\omega}^{ab}\tilde{\omega}_{ab} + \left[\frac{\tilde{h}^{ab}(\tilde{p}_{,b} + \tilde{\pi}^c_{b;c})}{\tilde{\mu} + \tilde{p}}\right]_{;a}. \tag{38}$$

This equation was derived in Eq. (88) of [23], see also [24]. In the multicomponent case, in the energy frame, combining Eqs. (31)–(34) we can derive

$$\left(\frac{\dot{\tilde{\tilde{\mu}}}_{(i)} + \tilde{\pi}^{ab}_{(i)}\tilde{\sigma}_{(i)ab} + \tilde{u}^a_{(i)}\tilde{I}_{(i)a}}{\tilde{\mu}_{(i)} + \tilde{p}_{(i)}}\right)^{\tilde{\cdot}} - \frac{1}{3} \times \left(\frac{\dot{\tilde{\tilde{\mu}}}_{(i)} + \tilde{\pi}^{ab}_{(i)}\tilde{\sigma}_{(i)ab} + \tilde{u}^a_{(i)}\tilde{I}_{(i)a}}{\tilde{\mu}_{(i)} + \tilde{p}_{(i)}}\right)^2 = -4\pi G(\tilde{\mu} - 3\tilde{p} - 2\tilde{T}_{ab}\tilde{u}^a_{(i)}\tilde{u}^b_{(i)}) - \Lambda + \tilde{\sigma}^{ab}_{(i)}\tilde{\sigma}_{(i)ab} - \tilde{\omega}^{ab}_{(i)}\tilde{\omega}_{(i)ab} + \left[\frac{\tilde{h}^{ab}_{(i)}(\tilde{p}_{(i),b} + \tilde{\pi}_{(i)b;c}^{\ \ \ \ \ \ c} - \tilde{I}_{(i)b})}{\tilde{\mu}_{(i)} + \tilde{p}_{(i)}}\right]_{;a}. \tag{39}$$

In [25] Langlois and Vernizzi derived a simple covariant relation which leads to one of the conserved variable in the large-scale limit. These authors introduced

$$\tilde{\zeta}_a \equiv \tilde{h}^b_a\left[\tilde{\alpha}_{,b} + \frac{\tilde{\mu}_{,b}}{3(\tilde{\mu} + \tilde{p})}\right], \qquad \dot{\tilde{\tilde{\alpha}}} \equiv \tilde{\alpha}_{,a}\tilde{u}^a \equiv \frac{1}{3}\tilde{\theta}. \tag{40}$$

Using only the energy conservation in Eq. (35) we can derive

$$\mathcal{L}_{\tilde{u}}\tilde{\zeta}_a = -\frac{\tilde{\theta}}{3(\tilde{\mu} + \tilde{p})}\left[\tilde{p}_{,a} + \frac{\dot{\tilde{\tilde{p}}}}{(\tilde{\mu} + \tilde{p})\tilde{\theta}}\tilde{\mu}_{,a}\right] - \left[\frac{\tilde{u}_a\tilde{\pi}^{bc}\tilde{\sigma}_{bc}}{3(\tilde{\mu} + \tilde{p})}\right]^{\tilde{\cdot}} - \frac{(\tilde{\pi}^{bc}\tilde{\sigma}_{bc})_{,a}}{3(\tilde{\mu} + \tilde{p})} + \frac{\tilde{\mu}_{,a}\tilde{\pi}^{bc}\tilde{\sigma}_{bc}}{3(\tilde{\mu} + \tilde{p})^2}, \tag{41}$$

where $\mathcal{L}_{\tilde{u}}$ is a Lie derivative along $\tilde{u}_a$ with $\mathcal{L}_{\tilde{u}}\tilde{\zeta}_a \equiv \tilde{\zeta}_{a;b}\tilde{u}^b + \tilde{\zeta}^b\tilde{u}_{b;a}$. These equations are valid in a single-component fluid, or in multiple component fluids for the collective fluid variables. For vanishing anisotropic pressure, we have the Langlois-Vernizzi relation [25]

$$\mathcal{L}_{\tilde{u}}\tilde{\zeta}_a = -\frac{\tilde{\theta}}{3(\tilde{\mu}+\tilde{p})}\tilde{h}_a^b\left(\tilde{p}_{,b} - \frac{\dot{\tilde{\tilde{p}}}}{\dot{\tilde{\tilde{\mu}}}}\tilde{\mu}_{,b}\right), \quad (42)$$

which is valid for a single-component fluid; in the case of a multicomponent situation, vanishing anisotropic stress of an individual component does not imply vanishing anisotropic stress of the collective component, see Eq. (142). We can easily extend the relation to the individual fluid component as follows. We introduce

$$\tilde{\zeta}_{(i)a} \equiv \tilde{h}_{(i)a}{}^{b}\left[\tilde{\alpha}_{(i),b} + \frac{\tilde{\mu}_{(i),b}}{3(\tilde{\mu}_{(i)}+\tilde{p}_{(i)})}\right],$$
$$\dot{\tilde{\tilde{\alpha}}}_{(i)} \equiv \tilde{\alpha}_{(i),a}\tilde{u}^a_{(i)} \equiv \frac{1}{3}\tilde{\theta}_{(i)}. \quad (43)$$

Using only Eq. (31) we can derive

$$\mathcal{L}_{\tilde{u}_{(i)}}\tilde{\zeta}_{(i)a} = -\frac{\tilde{\theta}_{(i)}}{3(\tilde{\mu}_{(i)}+\tilde{p}_{(i)})} \times\left[\tilde{p}_{(i),a} + \frac{\dot{\tilde{\tilde{p}}}_{(i)}}{(\tilde{\mu}_{(i)}+\tilde{p}_{(i)})\tilde{\theta}_{(i)}}\tilde{\mu}_{(i),a}\right] - \left[\frac{\tilde{u}_{(i)a}(\tilde{\pi}^{bc}_{(i)}\tilde{\sigma}_{(i)bc} + \tilde{u}^b_{(i)}\tilde{I}_{(i)b})}{3(\tilde{\mu}_{(i)}+\tilde{p}_{(i)})}\right]^{\dot{\tilde{}}} - \frac{(\tilde{\pi}^{bc}_{(i)}\tilde{\sigma}_{(i)bc} + \tilde{u}^b_{(i)}\tilde{I}_{(i)b})_{,a}}{3(\tilde{\mu}_{(i)}+\tilde{p}_{(i)})} + \frac{\tilde{\mu}_{(i),a}(\tilde{\pi}^{bc}_{(i)}\tilde{\sigma}_{(i)bc} + \tilde{u}^b_{(i)}\tilde{I}_{(i)b})}{3(\tilde{\mu}_{(i)}+\tilde{p}_{(i)})^2}. \quad (44)$$

Thus, for vanishing anisotropic pressure and direct interactions among fluids, i.e., $\tilde{\pi}_{(i)ab} = 0 = \tilde{I}_{(i)a}$, we have

$$\mathcal{L}_{\tilde{u}_{(i)}}\tilde{\zeta}_{(i)a} = -\frac{\tilde{\theta}_{(i)}}{3(\tilde{\mu}+\tilde{p})}\tilde{h}_{(i)a}{}^{b}\left(\tilde{p}_{(i),b} - \frac{\dot{\tilde{\tilde{p}}}_{(i)}}{\dot{\tilde{\tilde{\mu}}}_{(i)}}\tilde{\mu}_{(i),b}\right). \quad (45)$$

Applications of these compact relations to large-scale conservation properties to the second order will be studied in Sec. XI.

B. ADM equations

The ADM formulation [26] is presented in Eqs. (2)–(13), (47), and (48) of [18]. Interpretation of the ADM fluid quantities in Eqs. (45) and (46) of [18] was based on the normal-frame fluid quantities; for relations to the energy-frame fluid quantities, see Eq. (55) below. The ADM fluid quantities of the individual component are introduced as

$$E_{(i)} \equiv \tilde{n}_a\tilde{n}_b\tilde{T}^{ab}_{(i)} = N^2\tilde{T}^{00}_{(i)},$$
$$J_{(i)\alpha} \equiv -\tilde{n}_b\tilde{T}_{(i)\alpha}{}^{b} = N\tilde{T}_{(i)\alpha}{}^{0}, \qquad S_{(i)\alpha\beta} \equiv \tilde{T}_{(i)\alpha\beta}, \quad (46)$$
$$S_{(i)} \equiv h^{\alpha\beta}S_{(i)\alpha\beta}, \qquad \bar{S}_{(i)\alpha\beta} \equiv S_{(i)\alpha\beta} - \tfrac{1}{3}h_{\alpha\beta}S_{(i)}.$$

For the ADM fluid quantities of the collective component, we simply delete subindices (i) in the above equation. Equation (27) gives Eqs. (12), (13), (47), and (48) in [18]. Equations (10), (12), and (47) in [18] can be arranged as

$$\frac{1}{N}(\partial_0 - N^\alpha\partial_\alpha)K^\alpha_\alpha - \frac{1}{3}(K^\alpha_\alpha)^2 = 4\pi G(E+S) - \Lambda + \bar{K}^{\alpha\beta}\bar{K}_{\alpha\beta} - \frac{1}{N}N^{:\alpha}{}_{\alpha}, \quad (47)$$

$$K^\alpha_\alpha = \left(E + \frac{1}{3}S\right)^{-1}\left[\frac{1}{N}(\partial_0 - N^\alpha\partial_\alpha)E + \frac{1}{N^2}(N^2J^\alpha)_{:\alpha} - \bar{S}^{\alpha\beta}\bar{K}_{\alpha\beta}\right], \quad (48)$$

$$K^\alpha_\alpha = \left(E_{(i)} + \frac{1}{3}S_{(i)}\right)^{-1}\left[\frac{1}{N}(\partial_0 - N^\alpha\partial_\alpha)E_{(i)} + \frac{1}{N^2}(N^2J^\alpha_{(i)})_{:\alpha} - \bar{S}^{\alpha\beta}_{(i)}\bar{K}_{\alpha\beta} + \frac{1}{N}(\tilde{I}_{(i)0} - \tilde{I}_{(i)\alpha}N^\alpha)\right]. \quad (49)$$

Momentum conservation equations for the collective and individual components can be found in Eqs. (13) and (48) of [18]. By combining Eqs. (47)–(49) we can derive the ADM counterpart of the density perturbation equations in Eqs. (38) and (39). Indices of ADM quantities are based on $h_{\alpha\beta}$ as the metric with $h^{\alpha\beta}$ an inverse metric; these are $K_{\alpha\beta}$, J_α, $S_{\alpha\beta}$, etc.

V. SECOND-ORDER PERTURBATIONS

We use a metric convention in Eq. (49) of [18]

$$\tilde{g}_{00} \equiv -a^2(1+2A), \qquad \tilde{g}_{0\alpha} \equiv -a^2B_\alpha,$$
$$\tilde{g}_{\alpha\beta} \equiv a^2(g^{(3)}_{\alpha\beta} + 2C_{\alpha\beta}). \quad (50)$$

The subindex 0 indicates the conformal time η with $ad\eta \equiv cdt$. Indices of the perturbed-order quantities are based on $g^{(3)}_{\alpha\beta}$ as the metric with $g^{(3)\alpha\beta}$ an inverse metric; these are B_α, $C_{\alpha\beta}$, V_α, $\Pi_{\alpha\beta}$, Q_α, etc.

To the second order in perturbation we introduce the fluid four-vector of the collective component as

$$\tilde{u}^\alpha \equiv \frac{1}{a}V^\alpha. \quad (51)$$

The rest of the four-vector can be found in Eq. (53) of [18]; if we introduce $\tilde{u}^\alpha \equiv \bar{V}^\alpha\tilde{u}^0$, we have $V^\alpha = (1-A)\bar{V}^\alpha$. The fluid quantities of the collective component are introduced as

$$\tilde{\mu} \equiv \mu + \delta\mu, \qquad \tilde{p} \equiv p + \delta p, \qquad \tilde{\pi}_{\alpha\beta} \equiv a^2\Pi_{\alpha\beta},$$
$$\tilde{\pi}_{0\alpha} = -a^2\Pi_{\alpha\beta}V^\beta, \qquad \tilde{\pi}_{00} = 0, \quad (52)$$

where from $\tilde{\pi}^a_a \equiv 0$ we have

$$\Pi^\alpha_\alpha - 2C^{\alpha\beta}\Pi_{\alpha\beta} = 0. \quad (53)$$

The energy-momentum tensor of the collective component in the energy frame follows from Eq. (28) as

$$\begin{aligned}\tilde{T}^0_0 &= -\mu - \delta\mu - (\mu + p)(V^\alpha - B^\alpha)V_\alpha,\\ \tilde{T}^0_\alpha &= (\mu + p)(V_\alpha - B_\alpha - AV_\alpha + 2AB_\alpha + 2V^\beta C_{\alpha\beta})\\ &\quad + (\delta\mu + \delta p)(V_\alpha - B_\alpha) + \Pi_{\alpha\beta}(V^\beta - B^\beta),\\ \tilde{T}^\alpha_\beta &= (p + \delta p)\delta^\alpha_\beta + (\mu + p)V^\alpha(V_\beta - B_\beta)\\ &\quad + \Pi^\alpha_\beta - 2C^{\alpha\gamma}\Pi_{\beta\gamma}. \end{aligned} \tag{54}$$

From Eq. (46) we have the collective ADM fluid quantities based on the energy-frame fluid quantities

$$\begin{aligned} E &= \mu + \delta\mu + (\mu + p)(V^\alpha - B^\alpha)(V_\alpha - B_\alpha),\\ J_\alpha &= a[(\mu + p)(V_\alpha - B_\alpha + AB_\alpha + 2V^\beta C_{\alpha\beta})\\ &\quad + (\delta\mu + \delta p)(V_\alpha - B_\alpha) + \Pi_{\alpha\beta}(V^\beta - B^\beta)],\\ S &= 3(p + \delta p) + (\mu + p)(V^\alpha - B^\alpha)(V_\alpha - B_\alpha),\\ \bar{S}_{\alpha\beta} &= a^2\{\Pi_{\alpha\beta} + (\mu + p)[(V_\alpha - B_\alpha)(V_\beta - B_\beta)\\ &\quad - \tfrac{1}{3}g^{(3)}_{\alpha\beta}(V^\gamma - B^\gamma)(V_\gamma - B_\gamma)]\}. \end{aligned} \tag{55}$$

We can compare Eq. (55) with the ADM fluid quantities based on the normal-frame fluid quantities in Eq. (76) of [18]; in [18] the fluid quantities are based on the normal-frame vector. By taking $\tilde{u}_\alpha \equiv 0$, $\tilde{u}_a$ becomes the normal-frame vector $\tilde{n}_a$, see Eq. (54) in [18]. Based on the normal-frame vector, to the second order, the fluid quantities have contributions due to the frame choice; for example, even in the zero-pressure fluid, the perturbed pressure based on the normal frame does not necessarily vanish to the second order, see Eq. (56) below. By comparing Eq. (55) with Eq. (76) of [18] we have

$$\begin{aligned} \delta\mu^N &= \delta\mu + (\mu + p)(V^\alpha - B^\alpha)(V_\alpha - B_\alpha),\\ \delta p^N &= \delta p + \tfrac{1}{3}(\mu + p)(V^\alpha - B^\alpha)(V_\alpha - B_\alpha),\\ Q_\alpha &= (\mu + p)(V_\alpha - B_\alpha + AB_\alpha + 2V^\beta C_{\alpha\beta})\\ &\quad + (\delta\mu + \delta p)(V_\alpha - B_\alpha) + \Pi_{\alpha\beta}(V^\beta - B^\beta),\\ \Pi^N_{\alpha\beta} &= \Pi_{\alpha\beta} + (\mu + p)[(V_\alpha - B_\alpha)(V_\beta - B_\beta)\\ &\quad - \tfrac{1}{3}g^{(3)}_{\alpha\beta}(V^\gamma - B^\gamma)(V_\gamma - B_\gamma)]. \end{aligned} \tag{56}$$

These relations between the two frames are also presented in Eq. (87) of [18]. Thus, by replacing all fluid quantities in Eqs. (99)–(107) of [18] using Eq. (56) we have the equations in the energy frame. Using Q_α in Eq. (56), Eq. (54) gives

$$\tilde{T}^{\ 0}_{\alpha} = (1 - A)Q_\alpha. \tag{57}$$

As the fluid four-velocity of the collective component we can use either Q_α or $V_\alpha - B_\alpha$ related by Eq. (56).

For the individual component we have

$$\begin{aligned} \tilde{u}^{\ \alpha}_{(i)} &\equiv \frac{1}{a}V^\alpha_{(i)},\\ \tilde{u}^0_{(i)} &= \frac{1}{a}\Big(1 - A + \frac{3}{2}A^2 + \frac{1}{2}V^\alpha_{(i)}V_{(i)\alpha} - B^\alpha V_{(i)\alpha}\Big);\\ \tilde{u}_{(i)\alpha} &= a(V_{(i)\alpha} - B_\alpha + AB_\alpha + 2V^\beta_{(i)}C_{\alpha\beta}\Big),\\ \tilde{u}_{(i)0} &= -a\Big(1 + A - \frac{1}{2}A^2 + \frac{1}{2}V^\alpha_{(i)}V_{(i)\alpha}\Big), \end{aligned} \tag{58}$$

$$\begin{aligned} &\tilde{\mu}_{\ (i)} \equiv \mu_{(i)} + \delta\mu_{(i)}, \qquad \tilde{p}_{(i)} \equiv p_{(i)} + \delta p_{(i)},\\ &\tilde{\pi}_{(i)\alpha\beta} \equiv a^2\Pi_{(i)\alpha\beta}, \qquad \tilde{\pi}_{(i)0\alpha} = -a^2\Pi_{(i)\alpha\beta}V^\beta_{(i)},\\ &\tilde{\pi}_{(i)00} = 0, \end{aligned} \tag{59}$$

where from $\tilde{\pi}_{(i)a}{}^a \equiv 0$ we have

$$\Pi_{(i)\alpha}{}^\alpha - 2C^{\alpha\beta}\Pi_{(i)\alpha\beta} = 0. \tag{60}$$

For the energy-momentum tensor of an individual component in the energy frame, from Eq. (28) we have

$$\begin{aligned} \tilde{T}_{(i)0}{}^0 &= -\mu_{(i)} - \delta\mu_{(i)} - (\mu_{(i)} + p_{(i)})(V^\alpha_{(i)} - B^\alpha)V_{(i)\alpha},\\ \tilde{T}_{(i)\alpha}{}^0 &= (\mu_{(i)} + p_{(i)})(V_{(i)\alpha} - B_\alpha - AV_{(i)\alpha} + 2AB_\alpha\\ &\quad + 2V^\beta_{(i)}C_{\alpha\beta}) + (\delta\mu_{(i)} + \delta p_{(i)})(V_{(i)\alpha} - B_\alpha)\\ &\quad + \Pi_{(i)\alpha\beta}(V^\beta_{(i)} - B^\beta),\\ \tilde{T}_{(i)\beta}{}^\alpha &= (p_{(i)} + \delta p_{(i)})\delta^\alpha_\beta + (\mu_{(i)} + p_{(i)})V^\alpha_{(i)}(V_{(i)\beta} - B_\beta)\\ &\quad + \Pi_{(i)\beta}{}^\alpha - 2C^{\alpha\gamma}\Pi_{(i)\beta\gamma}. \end{aligned} \tag{61}$$

Using $\tilde{T}^a_b = \sum_j \tilde{T}_{(j)b}{}^a$, and the total fluid quantities in Eqs. (54) and (61) we have

$$\mu = \sum_j \mu_{(j)}, \qquad p = \sum_j p_{(j)} \tag{62}$$

for the background order fluid quantities, and

$$\begin{aligned} \delta\mu &= \sum_j [\delta\mu_{(j)} + (\mu_{(j)} + p_{(j)})(V^\alpha_{(j)} - B^\alpha)(V_{(j)\alpha} - V_\alpha)], \\ \delta p &= \sum_j [\delta p_{(j)} + \tfrac{1}{3}(\mu_{(j)} + p_{(j)})(V^\alpha_{(j)} - B^\alpha)(V_{(j)\alpha} - V_\alpha)], \\ (\mu + p)V_\alpha &= \sum_j [(\mu_{(j)} + p_{(j)})V_{(j)\alpha} + (\delta\mu_{(j)} + \delta p_{(j)})(V_{(j)\alpha} - V_\alpha) + \Pi_{(j)\alpha}{}^\beta (V_{(j)\beta} - V_\beta)], \\ \Pi^\alpha_\beta &= \sum_j \{\Pi_{(j)\beta}{}^\alpha + (\mu_{(j)} + p_{(j)})[(V^\alpha_{(j)} - B^\alpha)(V_{(j)\beta} - V_\beta) - \tfrac{1}{3}\delta^\alpha_\beta (V^\gamma_{(j)} - B^\gamma)(V_{(j)\gamma} - V_\gamma)]\} \end{aligned} \tag{63}$$

for perturbed-order fluid quantities to the second order. From Eq. (46) we have the ADM fluid quantities based on the energy-frame fluid quantities

$$\begin{aligned} E_{(i)} &= \mu_{(i)} + \delta\mu_{(i)} + (\mu_{(i)} + p_{(i)})(V^\alpha_{(i)} - B^\alpha)(V_{(i)\alpha} - B_\alpha), \\ J_{(i)\alpha} &= a[(\mu_{(i)} + p_{(i)})(V_{(i)\alpha} - B_\alpha + AB_\alpha + 2V^\beta_{(i)} C_{\alpha\beta}) + (\delta\mu_{(i)} + \delta p_{(i)})(V_{(i)\alpha} - B_\alpha) + \Pi_{(i)\alpha\beta}(V^\beta_{(i)} - B^\beta)], \\ S_{(i)} &= 3(p_{(i)} + \delta p_{(i)}) + (\mu_{(i)} + p_{(i)})(V^\alpha_{(i)} - B^\alpha)(V_{(i)\alpha} - B_\alpha), \\ \bar{S}_{(i)\alpha\beta} &= a^2\{\Pi_{(i)\alpha\beta} + (\mu_{(i)} + p_{(i)})[(V_{(i)\alpha} - B_\alpha)(V_{(i)\beta} - B_\beta) - \tfrac{1}{3} g^{(3)}_{\alpha\beta}(V^\gamma_{(i)} - B^\gamma)(V_{(i)\gamma} - B_\gamma)]\}. \end{aligned} \tag{64}$$

The kinematic quantities in the energy frame are presented in Eqs. (63)–(66) of [18]. In Eq. (33) we introduced kinematic quantities for the individual component. To the second order we can show

$$\begin{aligned} \tilde{\theta}_{(i)} &= 3\frac{a'}{a^2} - 3\frac{a'}{a^2}A + \frac{1}{a}C^{\alpha\prime}_\alpha + \frac{1}{a}V^\alpha_{(i)|\alpha} + \frac{9}{2}\frac{a'}{a^2}A^2 - 3\frac{a'}{a^2}B^\alpha V_{(i)\alpha} - \frac{1}{a}AC^{\alpha\prime}_\alpha + \frac{1}{a}V^\alpha_{(i)}(A_{,\alpha} + C^\beta_{\beta|\alpha}) + \frac{3}{2}\frac{a'}{a^2}V^\alpha_{(i)}V_{(i)\alpha} \\ &\quad + \frac{1}{a}(V^\alpha_{(i)} - B^\alpha)(V_{(i)\alpha} - B_\alpha)' - 2\frac{1}{a}C^{\alpha\beta}C'_{\alpha\beta}, \\ \tilde{\sigma}_{(i)\alpha\beta} &= a[V_{(i)(\alpha|\beta)} + C'_{\alpha\beta} - AC'_{\alpha\beta} + (V_{(i)(\alpha} - B_{(\alpha})(V_{(i)\beta)} - B_{\beta)})' + V_{(i)(\alpha}A_{,\beta)} + V^\gamma_{(i)}C_{\alpha\beta|\gamma} + 2V^\gamma_{(i)|(\alpha}C_{\beta)\gamma} \\ &\quad - \frac{2}{3}C_{\alpha\beta}(C^{\gamma\prime}_\gamma + V^\gamma_{(i)|\gamma})] - \frac{1}{3}g^{(3)}_{\alpha\beta}a[V^\gamma_{(i)|\gamma} + C^{\gamma\prime}_\gamma - AC^{\gamma\prime}_\gamma + (V^\gamma_{(i)} - B^\gamma)(V_{(i)\gamma} - B_\gamma)' + V^\gamma_{(i)}A_{,\gamma} + V^\gamma_{(i)}C^\delta_{\delta|\gamma} - 2C^{\gamma\delta}C'_{\gamma\delta}], \\ \tilde{\omega}_{(i)\alpha\beta} &= a(V_{(i)[\alpha} - B_{[\alpha} + AB_{[\alpha} + 2V^\gamma_{(i)}C_{\gamma[\alpha})_{|\beta]} - a(V_{(i)[\alpha} - B_{[\alpha})[(V_{(i)\beta]} - B_{\beta]})' + A_{,\beta]}], \\ \tilde{a}_{(i)\alpha} &= A_{,\alpha} + \frac{1}{a}\left[a(V_{(i)\alpha} - B_\alpha + AB_\alpha + 2V^\beta_{(i)}C_{\alpha\beta})\right]' - 2AA_{,\alpha} - A\frac{1}{a}[a(V_{(i)\alpha} - B_\alpha)]' + V^\beta_{(i)}[(V_{(i)\alpha} - B_\alpha)_{|\beta} + B_{\beta|\alpha}], \\ \tilde{\sigma}_{(i)\alpha 0} &= -V^\beta_{(i)}\tilde{\sigma}_{(i)\alpha\beta}, \quad \tilde{\sigma}_{(i)00} = 0; \quad \tilde{\omega}_{(i)\alpha 0} = -V^\beta_{(i)}\tilde{\omega}_{(i)\alpha\beta}, \quad \tilde{\omega}_{(i)00} = 0; \quad \tilde{a}_{(i)0} = -V^\alpha_{(i)}\tilde{a}_{(i)\alpha}, \end{aligned} \tag{65}$$

where a prime indicates the time derivative based on η.

The gauge transformation properties of the fluid quantities are presented in Eqs. (232)–(235) of [18] for the normal frame, and Eq. (238) of [18] for the energy frame. A prescription to get the gauge transformation properties for individual fluid quantities is also presented below Eq. (235) of [18]. Under the gauge transformation we have

$$\begin{aligned} \delta\hat{\mu}_{(i)} &= \delta\mu_{(i)} - (\mu'_{(i)} + \delta\mu'_{(i)})\xi^0 - \delta\mu_{(i),\alpha}\xi^\alpha + \frac{1}{2}\mu''_{(i)}\xi^0\xi^0 + \mu'_{(i)}(\xi^0\xi^{0\prime} + \xi^\alpha\xi^0{}_{,\alpha}), \\ \delta\hat{p}_{(i)} &= \delta p_{(i)} - (p'_{(i)} + \delta p'_{(i)})\xi^0 - \delta p_{(i),\alpha}\xi^\alpha + \frac{1}{2}p''_{(i)}\xi^0\xi^0 + p'_{(i)}(\xi^0\xi^{0\prime} + \xi^\alpha\xi^0{}_{,\alpha}), \\ \hat{V}_{(i)\alpha} - \hat{B}_\alpha + \hat{A}\hat{B}_\alpha + 2\hat{V}^\beta_{(i)}\hat{C}_{\alpha\beta} &= V_{(i)\alpha} - B_\alpha + AB_\alpha + 2V^\beta_{(i)}C_{\alpha\beta} + \xi^0{}_{,\alpha} - (V_{(i)\alpha} - B_\alpha)'\xi^0 - \frac{a'}{a}(V_{(i)\alpha} - B_\alpha)\xi^0 \\ &\quad - (V_{(i)\beta} - B_\beta)\xi^\beta{}_{,\alpha} - (V_{(i)\alpha} - B_\alpha)_{,\beta}\xi^\beta + \left(A - \xi^{0\prime} - \frac{a'}{a}\xi^0\right)\xi^0{}_{,\alpha} - \xi^\beta{}_{,\alpha}\xi^0{}_{,\beta} \\ &\quad - \xi^0\xi^{0\prime}{}_{,\alpha} - \xi^\beta\xi^0{}_{,\alpha\beta}, \\ \hat{\Pi}_{(i)\alpha\beta} &= \Pi_{(i)\alpha\beta} - \left(\Pi'_{(i)\alpha\beta} + 2\frac{a'}{a}\Pi_{(i)\alpha\beta}\right)\xi^0 - \Pi_{(i)\alpha\beta,\gamma}\xi^\gamma - 2\Pi_{(i)\gamma(\alpha}\xi^\gamma{}_{,\beta)}. \end{aligned} \tag{66}$$

A. Basic equations in the energy frame

The basic set of equations with fluid quantities based on the normal frame is presented in Eqs. (99)–(107) of [18]. By replacing the fluid quantities using Eq. (56) we can recover the equations with fluid quantities based on the energy frame. For convenience, in the following we present the complete set of equations with fluid quantities in the energy frame. These equations are written without taking any gauge conditions yet, thus in a sort of gauge-ready form. To the linear order this method was suggested by Bardeen [14,27].

Definition of δK ($K^\alpha_\alpha \equiv \bar{K} + \delta K$):

$$\bar{K} + 3H + \delta K - 3HA + \dot{C}^\alpha_\alpha + \frac{1}{a}B^\alpha{}_{|\alpha} = -A\left(\frac{9}{2}HA - \dot{C}^\alpha_\alpha - \frac{1}{a}B^\alpha{}_{|\alpha}\right) + \frac{3}{2}HB^\alpha B_\alpha + \frac{1}{a}B^\alpha(2C^\beta_{\alpha|\beta} - C^\beta_{\beta|\alpha}) + 2C^{\alpha\beta}\left(\dot{C}_{\alpha\beta} + \frac{1}{a}B_{\alpha|\beta}\right) \equiv n_0. \tag{67}$$

Energy constraint equation:

$$16\pi G\mu + 2\Lambda - 6H^2 - \frac{6K}{a^2} + 16\pi G\delta\mu + 4H\delta K - \frac{1}{a^2}(2C^{\beta|\alpha}_\alpha{}_\beta - 2C^{\alpha|\beta}_\alpha{}_\beta - 4KC^\alpha_\alpha)$$
$$= \frac{2}{3}\delta K^2 - 16\pi G(\mu + p)(V^\alpha - B^\alpha)(V_\alpha - B_\alpha) - \left(\dot{C}_{\alpha\beta} + \frac{1}{a}B_{(\alpha|\beta)}\right)\left(\dot{C}^{\alpha\beta} + \frac{1}{a}B^{\alpha|\beta}\right) + \frac{1}{3}\left(\dot{C}^\alpha_\alpha + \frac{1}{a}B^\alpha{}_{|\alpha}\right)^2$$
$$+ \frac{1}{a^2}[4C^{\alpha\beta}(-C^\gamma_{\alpha|\beta\gamma} - C^\gamma_{\alpha|\gamma\beta} + C_{\alpha\beta}{}^{|\gamma}{}_\gamma + C^\gamma_{\gamma|\alpha\beta}) + 8KC^\alpha_\gamma C^\gamma_\alpha - (2C^\gamma_{\beta|\gamma} - C^\gamma_{\gamma|\beta})(2C^{\alpha\beta}{}_{|\alpha} - C^{\alpha|\beta}_\alpha)$$
$$+ C^{\alpha\beta|\gamma}(3C_{\alpha\beta|\gamma} - 2C_{\alpha\gamma|\beta})] \equiv n_1. \tag{68}$$

Momentum constraint equation:

$$\left[\dot{C}^\beta_\alpha + \frac{1}{2a}(B^\beta{}_{|\alpha} + B_\alpha{}^{|\beta})\right]_{|\beta} - \frac{1}{3}\left(\dot{C}^\gamma_\gamma + \frac{1}{a}B^\gamma{}_{|\gamma}\right)_{,\alpha} + \frac{2}{3}\delta K_{,\alpha} + 8\pi Ga(\mu + p)(V_\alpha - B_\alpha + AB_\alpha + 2V^\beta C_{\alpha\beta})$$
$$= -\frac{2}{3}A\delta K_{,\alpha} - 8\pi Ga[(\delta\mu + \delta p)(V_\alpha - B_\alpha) + (\mu + p)A(V_\alpha - B_\alpha) + \Pi_{\alpha\beta}(V^\beta - B^\beta)]$$
$$+ A_{,\beta}\left[\dot{C}^\beta_\alpha + \frac{1}{2a}(B^\beta{}_{|\alpha} + B_\alpha{}^{|\beta})\right] + (2C^\beta_{\gamma|\beta} - C^\beta_{\beta|\gamma})\left[\dot{C}^\gamma_\alpha + \frac{1}{2a}(B_\alpha{}^{|\gamma} + B^\gamma{}_{|\alpha})\right] + 2C^{\beta\gamma}\left(\dot{C}_{\alpha\gamma} + \frac{1}{a}B_{(\alpha|\gamma)}\right)_{|\beta}$$
$$+ \frac{1}{a}[B_\gamma(C^{\beta\gamma}{}_{|\alpha} + C^{\gamma|\beta}_\alpha - C^{\beta|\gamma}_\alpha)]_{|\beta} + \frac{1}{3}C^\gamma_{\beta|\alpha}\left(\dot{C}^\beta_\gamma + \frac{1}{a}B^\beta{}_{|\gamma}\right) - \frac{1}{3}\left\{A_{,\alpha}\left(\dot{C}^\gamma_\gamma + \frac{1}{a}B^\gamma{}_{|\gamma}\right) + 2C^{\gamma\delta}\left(\dot{C}_{\gamma\delta} + \frac{1}{a}B_{\gamma|\delta}\right)_{|\alpha}\right.$$
$$\left.+ \frac{1}{a}[B^\delta(2C^\gamma_{\delta|\gamma} - C^\gamma_{\gamma|\delta})]_{|\alpha}\right\} \equiv n_{2\alpha}. \tag{69}$$

Trace of the ADM propagation equation:

$$-[3\dot{H} + 3H^2 + 4\pi G(\mu + 3p) - \Lambda] + \delta\dot{K} + 2H\delta K - 4\pi G(\delta\mu + 3\delta p) + \left(3\dot{H} + \frac{\Delta}{a^2}\right)A$$
$$= A\delta\dot{K} - \frac{1}{a}\delta K_{,\alpha}B^\alpha + \frac{1}{3}\delta K^2 + 8\pi G(\mu + p)(V^\alpha - B^\alpha)(V_\alpha - B_\alpha) + \frac{3}{2}\dot{H}(3A^2 - B^\alpha B_\alpha)$$
$$+ \frac{1}{a^2}\left[2A\Delta A + A^{,\alpha}A_{,\alpha} - \frac{1}{2}\Delta(B^\alpha B_\alpha) + A^{,\alpha}(2C^\beta_{\alpha|\beta} - C^\beta_{\beta|\alpha}) + 2C^{\alpha\beta}A_{,\alpha|\beta}\right]$$
$$+ \left(\dot{C}^{\alpha\beta} + \frac{1}{a}B^{(\alpha|\beta)}\right)\left(\dot{C}_{\alpha\beta} + \frac{1}{a}B_{\alpha|\beta}\right) - \frac{1}{3}\left(\dot{C}^\alpha_\alpha + \frac{1}{a}B^\alpha{}_{|\alpha}\right)^2 \equiv n_3. \tag{70}$$

Tracefree ADM propagation equation:

$$
\begin{aligned}
&\Big[\dot C^\alpha_\beta + \frac{1}{2a}(B^\alpha{}_{|\beta} + B_\beta{}^{|\alpha})\Big]^{\cdot} + 3H\Big[\dot C^\alpha_\beta + \frac{1}{2a}(B^\alpha{}_{|\beta} + B_\beta{}^{|\alpha})\Big] - \frac{1}{a^2}A^{|\alpha}{}_\beta - \frac{1}{3}\delta^\alpha_\beta\Big[\Big(\dot C^\gamma_\gamma + \frac{1}{a}B^\gamma{}_{|\gamma}\Big)^{\cdot} + 3H\Big(\dot C^\gamma_\gamma + \frac{1}{a}B^\gamma{}_{|\gamma}\Big) \\
&\quad - \frac{1}{a^2}A^{|\gamma}{}_\gamma\Big] + \frac{1}{a^2}\Big[C^{\alpha\gamma}{}_{|\beta\gamma} + C^{\gamma|\alpha}_{\beta}{}_{\gamma} - C^{\alpha|\gamma}_{\beta}{}_{\gamma} - C^{\gamma|\alpha}_{\gamma}{}_{\beta} - 4KC^\alpha_\beta - \frac{1}{3}\delta^\alpha_\beta(2C^{\delta|\gamma}_{\gamma}{}_{\delta} - 2C^{\gamma|\delta}_{\gamma}{}_{\delta} - 4KC^\gamma_\gamma)\Big] - 8\pi G\Pi^\alpha_\beta \\
&\quad = \Big\{\Big[\dot C^\alpha_\beta + \frac{1}{2a}(B^\alpha{}_{|\beta} + B_\beta{}^{|\alpha})\Big]A + 2C^{\alpha\gamma}\Big(\dot C_{\beta\gamma} + \frac{1}{a}B_{(\beta|\gamma)}\Big) + \frac{1}{a}B_\gamma(C^{\alpha\gamma}{}_{|\beta} + C^{\gamma|\alpha}_\beta - C^{\alpha|\gamma}_\beta)\Big\}^{\cdot} \\
&\qquad + 3H\Big\{\Big[\dot C^\alpha_\beta + \frac{1}{2a}(B^\alpha{}_{|\beta} + B_\beta{}^{|\alpha})\Big]A + 2C^{\alpha\gamma}\Big(\dot C_{\beta\gamma} + \frac{1}{a}B_{(\beta|\gamma)}\Big) + \frac{1}{a}B_\gamma(C^{\alpha\gamma}{}_{|\beta} + C^{\gamma|\alpha}_\beta - C^{\alpha|\gamma}_\beta)\Big\} \\
&\qquad + \Big[\dot C^\alpha_\beta + \frac{1}{2a}(B^\alpha{}_{|\beta} + B_\beta{}^{|\alpha})\Big]^{\cdot}A - \frac{1}{a}\Big[\dot C^\alpha_\beta + \frac{1}{2a}(B^\alpha{}_{|\beta} + B_\beta{}^{|\alpha})\Big]_{|\gamma}B^\gamma + \delta K\Big[\dot C^\alpha_\beta + \frac{1}{2a}(B^\alpha{}_{|\beta} + B_\beta{}^{|\alpha})\Big] \\
&\qquad + \frac{1}{a^2}\Big[-AA^{|\alpha}{}_\beta + \frac{1}{2}(-A^2 + B^\gamma B_\gamma)^{|\alpha}{}_\beta - 2C^{\alpha\gamma}A_{,\beta|\gamma} - (C^{\alpha\gamma}{}_{|\beta} + C^{\gamma|\alpha}_\beta - C^{\alpha|\gamma}_\beta)A_{,\gamma}\Big] - \frac{1}{3}\delta^\alpha_\beta\Big\{\Big[\Big(\dot C^\gamma_\gamma + \frac{1}{a}B^\gamma{}_{|\gamma}\Big)A \\
&\qquad + 2C^{\gamma\delta}\Big(\dot C_{\gamma\delta} + \frac{1}{a}B_{\gamma|\delta}\Big) + \frac{1}{a}B^\delta(2C^\gamma_{\delta|\gamma} - C^\gamma_{\gamma|\delta})\Big]^{\cdot} + 3H\Big[\Big(\dot C^\gamma_\gamma + \frac{1}{a}B^\gamma{}_{|\gamma}\Big)A + 2C^{\gamma\delta}\Big(\dot C_{\gamma\delta} + \frac{1}{a}B_{\gamma|\delta}\Big) \\
&\qquad + \frac{1}{a}B^\delta(2C^\gamma_{\delta|\gamma} - C^\gamma_{\gamma|\delta})\Big] + \Big(\dot C^\gamma_\gamma + \frac{1}{a}B^\gamma{}_{|\gamma}\Big)^{\cdot}A - \frac{1}{a}\Big(\dot C^\gamma_\gamma + \frac{1}{a}B^\gamma{}_{|\gamma}\Big)_{|\delta}B^\delta + \delta K\Big(\dot C^\gamma_\gamma + \frac{1}{a}B^\gamma{}_{|\gamma}\Big) + \frac{1}{a^2}\Big[-AA^{|\gamma}{}_\gamma \\
&\qquad + \frac{1}{2}(-A^2 + B^\delta B_\delta)^{|\gamma}{}_\gamma - 2C^{\gamma\delta}A_{,\gamma|\delta} - (2C^{\gamma\delta}{}_{|\gamma} - C^{\gamma|\delta}_\gamma)A_{,\delta}\Big]\Big\} + \frac{1}{a}B^\alpha{}_{|\gamma}\Big[\dot C^\gamma_\beta + \frac{1}{2a}(B^\gamma{}_{|\beta} + B_\beta{}^{|\gamma})\Big] \\
&\qquad - \frac{1}{a}B^\gamma{}_{|\beta}\Big[\dot C^\alpha_\gamma + \frac{1}{2a}(B^\alpha{}_{|\gamma} + B_\gamma{}^{|\alpha})\Big] + \frac{1}{a^2}\Big\{2C^{\gamma\delta}(C^\alpha_{\delta|\beta\gamma} + C_{\delta\beta}{}^{|\alpha}{}_\gamma - C^\alpha_{\beta|\delta\gamma} - C_{\delta\gamma}{}^{|\alpha}{}_\beta) + 2C^{\alpha\gamma}(C^\delta_{\gamma|\beta\delta} + C^\delta_{\beta|\gamma\delta} \\
&\qquad - C_{\beta\gamma}{}^{|\delta}{}_\delta - C^\delta_{\delta|\gamma\beta}) - 8KC^\alpha_\gamma C^\gamma_\beta + (2C^\gamma_{\delta|\gamma} - C^\gamma_{\gamma|\delta})(C^{\alpha\delta}{}_{|\beta} + C^{\delta|\alpha}_\beta - C^{\alpha|\delta}_\beta) - C_{\gamma\delta|\beta}C^{\gamma\delta|\alpha} + 2C^{\alpha\gamma|\delta}(C_{\beta\delta|\gamma} - C_{\beta\gamma|\delta}) \\
&\qquad - \frac{1}{3}\delta^\alpha_\beta[4C^{\gamma\delta}(C^\epsilon_{\gamma|\delta\epsilon} + C^\epsilon_{\gamma|\epsilon\delta} - C_{\gamma\delta}{}^{|\epsilon}{}_\epsilon - C^\epsilon_{\epsilon|\gamma\delta}) - 8KC^\delta_\gamma C^\gamma_\delta + (2C^\epsilon_{\delta|\epsilon} - C^\epsilon_{\epsilon|\delta})(2C^{\gamma\delta}{}_{|\gamma} - C^{\gamma|\delta}_\gamma) \\
&\qquad + C^{\gamma\delta|\epsilon}(2C_{\gamma\epsilon|\delta} - 3C_{\gamma\delta|\epsilon})\Big]\Big\} - 16\pi G C^{\alpha\gamma}\Pi_{\beta\gamma} + 8\pi G(\mu + p)\Big[(V^\alpha - B^\alpha)(V_\beta - B_\beta) \\
&\qquad - \frac{1}{3}\delta^\alpha_\beta(V^\gamma - B^\gamma)(V_\gamma - B_\gamma)\Big] \equiv n_{4\beta}{}^\alpha. \approx
\end{aligned}
\tag{71}
$$

Energy conservation equation:

$$
\begin{aligned}
&[\dot\mu + 3H(\mu + p)] + \delta\dot\mu + 3H(\delta\mu + \delta p) - (\mu + p)(\delta K - 3HA) + \frac{1}{a}(\mu + p)[V^\alpha - B^\alpha + AB^\alpha + 2V^\beta C^\alpha_\beta]_{|\alpha} \\
&\quad = -\frac{1}{a}\delta\mu_{,\alpha}B^\alpha + (\delta\mu + \delta p)(\delta K - 3HA) + (\mu + p)A\delta K + \frac{3}{2}H(\mu + p)(A^2 - B^\alpha B_\alpha) \\
&\qquad - \frac{1}{a^4}[a^4(\mu + p)(V^\alpha - B^\alpha)(V_\alpha - B_\alpha)]^{\cdot} - \frac{1}{a}[(\delta\mu + \delta p)(V^\alpha - B^\alpha) + \Pi^{\alpha\beta}(V_\beta - B_\beta)]_{|\alpha} \\
&\qquad + \frac{1}{a}(\mu + p)\{-A(V^\alpha - B^\alpha)_{|\alpha} - 2A_{,\alpha}(V^\alpha - B^\alpha) + 2[C^{\alpha\beta}(V_\beta - B_\beta)]_{|\alpha} - C^\alpha_{\alpha|\beta}(V^\beta - B^\beta)\} \\
&\qquad - \Pi^{\alpha\beta}\Big(\dot C_{\alpha\beta} + \frac{1}{a}B_{\alpha|\beta}\Big) \equiv n_5.
\end{aligned}
\tag{72}
$$

Momentum conservation equation:

$$
\begin{aligned}
&\frac{1}{a^4}[a^4(\mu + p)(V_\alpha - B_\alpha + AB_\alpha + 2V^\beta C_{\alpha\beta})]^{\cdot} + \frac{1}{a}(\mu + p)A_{,\alpha} + \frac{1}{a}(\delta p_{,\alpha} + \Pi^\beta_{\alpha|\beta}) \\
&\quad = (\mu + p)(\delta K - 3HA)(V_\alpha - B_\alpha) - \frac{1}{a^4}\{a^4[(\delta\mu + \delta p)(V_\alpha - B_\alpha) + \Pi^\beta_\alpha(V_\beta - B_\beta)]\}^{\cdot} + \frac{1}{a}\{-(\delta p_{,\alpha} + \Pi^\beta_{\alpha|\beta})A \\
&\qquad - (\delta\mu + \delta p)A_{,\alpha} - (\mu + p)[-AA_{,\alpha} + B_{\beta|\alpha}V^\beta + (V_\alpha - B_\alpha)_{|\beta}V^\beta + (V_\alpha - B_\alpha)(V^\beta - B^\beta)_{|\beta}] + 2(C^{\beta\gamma}\Pi_{\alpha\gamma})_{|\beta} \\
&\qquad - C^\beta_{\beta|\gamma}\Pi^\gamma_\alpha + C^\gamma_{\beta|\alpha}\Pi^\beta_\gamma - A_{,\beta}\Pi^\beta_\alpha\} \equiv n_{6\alpha}.
\end{aligned}
\tag{73}
$$

In the multicomponent situation we additionally have the energy and the momentum conservation of the individual component. Using the energy-frame fluid quantities, Eqs. (106) and (107) in [18] become

$$\left[\dot\mu_{(i)} + 3H(\mu_{(i)} + p_{(i)}) + \frac{1}{a}I_{(i)0}\right] + \delta\dot\mu_{(i)} + 3H(\delta\mu_{(i)} + \delta p_{(i)}) - (\mu_{(i)} + p_{(i)})(\delta K - 3HA)$$
$$+ \frac{1}{a}(\mu_{(i)} + p_{(i)})[V^\alpha_{(i)} - B^\alpha + AB^\alpha + 2V^\beta_{(i)}C^\alpha_\beta]_{|\alpha} + \frac{1}{a}\delta I_{(i)0}$$
$$= -\frac{1}{a}\delta\mu_{(i),\alpha}B^\alpha + (\delta\mu_{(i)} + \delta p_{(i)})(\delta K - 3HA) + (\mu_{(i)} + p_{(i)})A\delta K + \frac{3}{2}H(\mu_{(i)} + p_{(i)})(A^2 - B^\alpha B_\alpha)$$
$$- \frac{1}{a^4}[a^4(\mu_{(i)} + p_{(i)})(V^\alpha_{(i)} - B^\alpha)(V_{(i)\alpha} - B_\alpha)]^\cdot - \frac{1}{a}[(\delta\mu_{(i)} + \delta p_{(i)})(V^\alpha_{(i)} - B^\alpha) + \Pi^{\alpha\beta}_{(i)}(V_{(i)\beta} - B_\beta)]_{|\alpha}$$
$$+ \frac{1}{a}(\mu_{(i)} + p_{(i)})\{-A(V^\alpha_{(i)} - B^\alpha)_{|\alpha} - 2A_{,\alpha}(V^\alpha_{(i)} - B^\alpha) + 2[C^{\alpha\beta}(V_{(i)\beta} - B_\beta)]_{|\alpha} - C^{\alpha|\beta}_\alpha(V_{(i)\beta} - B_\beta)\}$$
$$- \Pi^{\alpha\beta}_{(i)}\left(\dot C_{\alpha\beta} + \frac{1}{a}B_{\alpha|\beta}\right) - \frac{1}{a}\delta I_{(i)\alpha}B^\alpha \equiv n_{(i)5}, \tag{74}$$

$$\frac{1}{a^4}[a^4(\mu_{(i)} + p_{(i)})(V_{(i)\alpha} - B_\alpha + AB_\alpha + 2V^\beta_{(i)}C_{\alpha\beta})]^\cdot + \frac{1}{a}(\mu_{(i)} + p_{(i)})A_{,\alpha} + \frac{1}{a}(\delta p_{(i),\alpha} + \Pi^{\ \ \beta}_{(i)\alpha|\beta} - \delta I_{(i)\alpha})$$
$$= (\mu_{(i)} + p_{(i)})(\delta K - 3HA)(V_{(i)\alpha} - B_\alpha) - \frac{1}{a^4}\{a^4[(\delta\mu_{(i)} + \delta p_{(i)})(V_{(i)\alpha} - B_\alpha) + \Pi_{(i)\alpha}{}^\beta(V_{(i)\beta} - B_\beta)]\}^\cdot$$
$$+ \frac{1}{a}\{-(\delta p_{(i),\alpha} + \Pi^{\ \ \beta}_{(i)\alpha|\beta} - \delta I_{(i)\alpha})A - (\delta\mu_{(i)} + \delta p_{(i)})A_{,\alpha} - (\mu_{(i)} + p_{(i)})[-AA_{,\alpha} + B_{\beta|\alpha}V^\beta_{(i)} + (V_{(i)\alpha} - B_\alpha)_{|\beta}V^\beta_{(i)}$$
$$+ (V_{(i)\alpha} - B_\alpha)(V^\beta_{(i)} - B^\beta)_{|\beta}] + 2(C^{\beta\gamma}\Pi_{(i)\alpha\gamma})_{|\beta} - C^\beta_{\beta|\gamma}\Pi_{(i)\alpha}{}^\gamma + C^\gamma_{\beta|\alpha}\Pi_{(i)\gamma}{}^\beta - A_{,\beta}\Pi_{(i)\alpha}{}^\beta\Big\} \equiv n_{(i)6\alpha}. \tag{75}$$

By removing indices indicating the components in Eqs. (74) and (75) we recover equations for the collective component which coincide with the equations in a single-component situation in Eqs. (72) and (73).

To the background order, from Eqs. (70) and (74) we have

$$\dot\mu_{(i)} + 3H(\mu_{(i)} + p_{(i)}) = -\frac{c}{a}I_{(i)0}, \tag{76}$$

$$\dot\mu + 3H(\mu + p) = 0, \tag{77}$$

$$3\dot H + 3H^2 = -\frac{4\pi G}{c^2}\sum_j(\mu_{(j)} + 3p_{(j)}) + \Lambda c^2, \tag{78}$$

$$H^2 = \frac{8\pi G}{3c^2}\sum_j\mu_{(j)} - \frac{Kc^2}{a^2} + \frac{\Lambda c^2}{3}, \tag{79}$$

where we recovered the speed of light c. Dimensions are

$$\begin{gathered}[\tilde g_{ab}] = [\tilde g^{ab}] = [\tilde u_a] = 1, \qquad [\tilde T_{ab}] = [\mu],\\ [g^{(3)}_{\alpha\beta}] = 1, \qquad [a] = 1, \qquad [K] = [\Lambda] = L^{-2},\\ [\nabla] = L^{-1}, \qquad [\Delta] = L^{-2}, \qquad [G\varrho] = T^{-2},\\ [c] = LT^{-1}, \qquad [\eta] = L,\\ [p] = [\mu] = [\varrho c^2], \qquad [I_{(i)0}] = [\mu]L^{-1}.\end{gathered} \tag{80}$$

Equation (77) follows from the sum of Eq. (76) over components. Equation (79) follows from integrating Eq. (78) where the K term can be regarded as an integration constant; in Einstein's gravity the K term can be normalized, using a, as the sign of spatial curvature. Compared with the Newtonian background equations in Eq. (7), ignoring the direct interaction terms in Eq. (76), the presence of pressure terms in Eqs. (76)–(79) is the pure general relativistic effect. The cosmological constant Λ can be introduced by hand even in the Newtonian case.

B. Decomposition

We decompose the metric to three perturbation types

$$\begin{gathered}A \equiv \alpha, \qquad B_\alpha \equiv \beta_{,\alpha} + B^{(v)}_\alpha,\\ C_{\alpha\beta} \equiv \varphi g^{(3)}_{\alpha\beta} + \gamma_{,\alpha|\beta} + C^{(v)}_{(\alpha|\beta)} + C^{(t)}_{\alpha\beta},\end{gathered} \tag{81}$$

where superscripts (v) and (t) indicate the transverse vector-type, and transverse-tracefree tensor-type perturbations, respectively. We introduce

$$\chi \equiv a(\beta + c^{-1}a\dot\gamma), \qquad \Psi^{(v)}_\alpha \equiv B^{(v)}_\alpha + c^{-1}a\dot C^{(v)}_\alpha, \tag{82}$$

which are spatially gauge invariant to the linear order. We set

$$K^\alpha_\alpha \equiv -3H + \kappa. \tag{83}$$

Later, we will identify κ in the comoving gauge with the perturbed Newtonian velocity variable which will be an important step in our analysis, see Eqs. (127), (202), and (224).

For collective fluid quantities, we decompose

$$\tilde{u}_\alpha = a(V_\alpha - B_\alpha + AB_\alpha + 2V^\beta C_{\alpha\beta}) \equiv a v_\alpha \equiv a(-v_{,\alpha} + v_\alpha^{(v)}), \tag{84}$$

$$\Pi_{\alpha\beta} \equiv \frac{1}{a^2}\left(\Pi_{,\alpha|\beta} - \frac{1}{3} g^{(3)}_{\alpha\beta}\Delta\Pi\right) + \frac{1}{a}\Pi^{(v)}_{(\alpha|\beta)} + \Pi^{(t)}_{\alpha\beta}. \tag{85}$$

The perturbed fluid velocity variables v and $v_\alpha^{(v)}$ subtly differ from the ones introduced in [18]; see Sec. V C. For the individual fluid quantities, similarly, we have

$$\tilde{u}_{(i)\alpha} = a(V_{(i)\alpha} - B_\alpha + AB_\alpha + 2V^\beta_{(i)} C_{\alpha\beta}) \equiv a v_{(i)\alpha} \equiv a(-v_{(i),\alpha} + v_{(i)\alpha}^{(v)}), \tag{86}$$

$$\Pi_{(i)\alpha\beta} \equiv \frac{1}{a^2}\left(\Pi_{(i),\alpha|\beta} - \frac{1}{3} g^{(3)}_{\alpha\beta}\Delta\Pi_{(i)}\right) + \frac{1}{a}\Pi^{(v)}_{(i)(\alpha|\beta)} + \Pi^{(t)}_{(i)\alpha\beta}, \tag{87}$$

$$\delta I_{(i)\alpha} \equiv \delta I_{(i),\alpha} + \delta I^{(v)}_{(i)\alpha}.$$

For isotropic pressure we introduce

$$\delta p \equiv c_s^2 \delta\mu + e, \qquad c_s^2 \equiv \frac{\dot{p}}{\dot{\mu}}, \qquad w \equiv \frac{p}{\mu}. \tag{88}$$

The perturbation variable e is called an entropic perturbation. Defined in this way e is gauge invariant only to the linear order. To the second order, from Eq. (66) we can derive the following gauge-invariant combination

$$\delta p_{\delta\mu} \equiv e - \frac{\delta\mu}{\dot{\mu}}\left[\dot{e} + \frac{1}{2}(c_s^2)^\cdot \delta\mu\right]. \tag{89}$$

In our notation, $\delta p_{\delta\mu}$ is a gauge-invariant combination which is the same as δp in the $\delta\mu = 0$ slicing (temporal gauge) condition to the second order; as the spatial gauge we take $\gamma \equiv 0$ to the second order; for the derivation, see the prescription below Eq. (266) of [18]. In the multicomponent case, we similarly have

$$\delta p_{(i)\delta\mu_{(i)}} \equiv e_{(i)} - \frac{\delta\mu_{(i)}}{\dot{\mu}_{(i)}}\left[\dot{e}_{(i)} + \frac{1}{2}(c_{(i)}^2)^\cdot \delta\mu_{(i)}\right], \tag{90}$$

where

$$\delta p_{(i)} \equiv c_{(i)}^2 \delta\mu_{(i)} + e_{(i)}, \qquad c_{(i)}^2 \equiv \frac{\dot{p}_{(i)}}{\dot{\mu}_{(i)}}. \tag{91}$$

We have [28]

$$e = e_{\rm rel} + e_{\rm int}, \quad e_{\rm rel} \equiv \sum_j (c_{(j)}^2 - c_s^2)\delta\mu_{(j)}, \quad e_{\rm int} \equiv \sum_j e_{(j)}. \tag{92}$$

Equation (63) gives

$$\delta\mu = \sum_j [\delta\mu_{(j)} + (\mu_{(j)} + p_{(j)}) v^\alpha_{(j)}(v_{(j)\alpha} - v_\alpha)],$$
$$\delta p = \sum_j [\delta p_{(j)} + \tfrac{1}{3}(\mu_{(j)} + p_{(j)}) v^\alpha_{(j)}(v_{(j)\alpha} - v_\alpha)],$$
$$(\mu + p) v_\alpha = \sum_j [(\mu_{(j)} + p_{(j)}) v_{(j)\alpha} + (\delta\mu_{(j)} + \delta p_{(j)}) \times (v_{(j)\alpha} - v_\alpha) + \Pi_{(j)\alpha}{}^\beta (v_{(j)\beta} - v_\beta)],$$
$$\Pi^\alpha_\beta = \sum_j \{\Pi_{(j)\beta}{}^\alpha + (\mu_{(j)} + p_{(j)})[v^\alpha_{(j)}(v_{(j)\beta} - v_\beta) - \tfrac{1}{3}\delta^\alpha_\beta v^\gamma_{(j)}(v_{(j)\gamma} - v_\gamma)]\}. \tag{93}$$

Notice that even for the vanishing pressure and anisotropic stress of an individual component the pressure and anisotropic stress of a collective component do not vanish to the second order. By recovering c, dimensions of the variables are

$$\begin{aligned}
&[A] = [B_\alpha] = [C_{\alpha\beta}] = 1,\\
&[\alpha] = [\varphi] = [B^{(v)}_\alpha] = [\Psi^{(v)}_\alpha] = [C^{(t)}_{\alpha\beta}] = 1,\\
&[\beta] = [\chi] = [C^{(v)}_\alpha] = L, \qquad [\gamma] = L^2,\\
&[\kappa] = T^{-1}, \qquad [w] = [c_s^2] = 1,\\
&[\delta\mu] = [\delta p] = [e] = [\Pi_{\alpha\beta}] = [\Pi^{(t)}_{\alpha\beta}] = [\mu],\\
&[\delta] = [V_\alpha] = 1, \qquad [v] = L, \qquad [v^{(v)}_\alpha] = 1,\\
&[\Pi] = L^2[\mu], \qquad [\Pi^{(v)}_\alpha] = L[\mu].
\end{aligned} \tag{94}$$

Scalar-type perturbation equations can be derived from Eqs. (67)–(75)

$$\kappa - 3H\alpha + 3\dot{\varphi} + \frac{\Delta}{a^2}\chi = n_0, \tag{95}$$

$$4\pi G\delta\mu + H\kappa + \frac{\Delta + 3K}{a^2}\varphi = \frac{1}{4}n_1, \tag{96}$$

$$\kappa + \frac{\Delta + 3K}{a^2}\chi - 12\pi G(\mu + p)av = n_2 \equiv \frac{3}{2}\Delta^{-1}\nabla^\alpha n_{2\alpha}, \tag{97}$$

$$\dot{\kappa} + 2H\kappa - 4\pi G(\delta\mu + 3\delta p) + \left(3\dot{H} + \frac{\Delta}{a^2}\right)\alpha = n_3, \tag{98}$$

$$\dot{\chi} + H\chi - \varphi - \alpha - 8\pi G\Pi = n_4 \equiv \tfrac{3}{2}a^2(\Delta + 3K)^{-1}\Delta^{-1}\nabla^\alpha\nabla_\beta n_{4\alpha}{}^\beta, \tag{99}$$

$$\delta\dot{\mu} + 3H(\delta\mu + \delta p) - (\mu + p)\left(\kappa - 3H\alpha + \frac{\Delta}{a}v\right) = n_5, \tag{100}$$

$$\frac{[a^4(\mu+p)v]^{\cdot}}{a^4(\mu+p)} - \frac{1}{a}\alpha - \frac{1}{a(\mu+p)}\left(\delta p + \frac{2}{3}\frac{\Delta+3K}{a^2}\Pi\right) = n_6 \equiv -\frac{1}{\mu+p}\Delta^{-1}\nabla^\alpha n_{6\alpha}, \quad (101)$$

$$\delta\dot{\mu}_{(i)} + 3H(\delta\mu_{(i)} + \delta p_{(i)}) - (\mu_{(i)} + p_{(i)}) \times\left(\kappa - 3H\alpha + \frac{\Delta}{a}v_{(i)}\right) + \frac{1}{a}\delta I_{(i)0} = n_{5(i)}, \quad (102)$$

$$\frac{[a^4(\mu_{(i)}+p_{(i)})v_{(i)}]^{\cdot}}{a^4(\mu_{(i)}+p_{(i)})} - \frac{1}{a}\alpha - \frac{1}{a(\mu_{(i)}+p_{(i)})} \times\left(\delta p_{(i)} + \frac{2}{3}\frac{\Delta+3K}{a^2}\Pi_{(i)} - \delta I_{(i)}\right) = n_{6(i)} \equiv -\frac{1}{\mu_{(i)}+p_{(i)}}\Delta^{-1}\nabla^\alpha n_{6\alpha(i)}. \quad (103)$$

Equations for the vector-type perturbation follow from Eqs. (69), (71), (73), and (75)

$$\frac{\Delta+2K}{2a^2}\Psi^{(v)}_\alpha + 8\pi G(\mu+p)v^{(v)}_\alpha = \frac{1}{a}(n_{2\alpha} - \nabla_\alpha\Delta^{-1}\nabla^\beta n_{2\beta}) \equiv n^{(v)}_{2\alpha}, \quad (104)$$

$$\dot{\Psi}^{(v)}_\alpha + 2H\Psi^{(v)}_\alpha - 8\pi G\Pi^{(v)}_\alpha = 2a(\Delta+2K)^{-1}(\nabla_\beta n_{4_\alpha}{}^\beta - \nabla_\alpha\Delta^{-1}\nabla^\gamma\nabla_\beta n_{4_\gamma}{}^\beta) \equiv n^{(v)}_{4\alpha}, \quad (105)$$

$$\frac{[a^4(\mu+p)v^{(v)}_\alpha]^{\cdot}}{a^4(\mu+p)} + \frac{\Delta+2K}{2a^2}\frac{\Pi^{(v)}_\alpha}{\mu+p} = \frac{1}{\mu+p}(n_{6\alpha} - \nabla_\alpha\Delta^{-1}\nabla^\beta n_{6\beta}) \equiv n^{(v)}_{6\alpha}, \quad (106)$$

$$\frac{[a^4(\mu_{(i)}+p_{(i)})v^{(v)}_{(i)\alpha}]^{\cdot}}{a^4(\mu_{(i)}+p_{(i)})} + \frac{\Delta+2K}{2a^2}\frac{\Pi^{(v)}_{(i)\alpha}}{\mu_{(i)}+p_{(i)}} - \frac{1}{a}\frac{\delta I^{(v)}_{(i)\alpha}}{\mu_{(i)}+p_{(i)}} = \frac{1}{\mu_{(i)}+p_{(i)}}(n_{6(i)\alpha} - \nabla_\alpha\Delta^{-1}\nabla^\beta n_{6(i)\beta}) \equiv n^{(v)}_{6(i)\alpha}. \quad (107)$$

Equations for the tensor-type perturbation follow from Eq. (71)

$$\ddot{C}^{(t)}_{\alpha\beta} + 3H\dot{C}^{(t)}_{\alpha\beta} - \frac{\Delta-2K}{a^2}C^{(t)}_{\alpha\beta} - 8\pi G\Pi^{(t)}_{\alpha\beta} = n_{4\alpha\beta} - \frac{3}{2}\left(\nabla_\alpha\nabla_\beta - \frac{1}{3}g^{(3)}_{\alpha\beta}\Delta\right)(\Delta + 3K)^{-1}\Delta^{-1}\nabla^\gamma\nabla_\delta n_{4_\gamma}{}^\delta - 2\nabla_{(\alpha}(\Delta+2K)^{-1} \times(\nabla^\gamma n_{4\beta)\gamma} - \nabla_{\beta)}\Delta^{-1}\nabla^\gamma\nabla_\delta n_{4_\gamma}{}^\delta) \equiv n^{(t)}_{4\alpha\beta}. \quad (108)$$

In order to derive Eqs. (99), (105), and (108) it is convenient to show

$$\frac{1}{a^2}\left(\nabla_\alpha\nabla_\beta - \frac{1}{3}g^{(3)}_{\alpha\beta}\Delta\right)(\dot{\chi} + H\chi - \varphi - \alpha - 8\pi G\Pi) + \frac{1}{a^3}(a^2\Psi^{(v)}_{(\alpha|\beta)})^{\cdot} - 8\pi G\frac{1}{a}\Pi^{(v)}_{(\alpha|\beta)} + \ddot{C}^{(t)}_{\alpha\beta} + 3H\dot{C}^{(t)}_{\alpha\beta} - \frac{\Delta-2K}{a^2}C^{(t)}_{\alpha\beta} - 8\pi G\Pi^{(t)}_{\alpha\beta} = n_{4\alpha\beta}, \quad (109)$$

which follows from Eq. (71). Quadratic combinations of linear-order perturbation variables of all three types of perturbations contribute to all three types of perturbation to the second order. In this way, in the perturbation approach the three perturbation types are coupled to the second order.

C. Comoving gauge and irrotational condition

In Eq. (180) of [18] we introduced

$$\tilde{T}^0_\alpha = (1-A)Q_\alpha, \qquad Q_\alpha \equiv (\mu+p)(-\bar{v}_{,\alpha} + \bar{v}^{(v)}_\alpha), \quad (110)$$

where we put overbars to $\bar{v}$ and $\bar{v}^{(v)}_\alpha$ in order to distinguish these from our new notations to be used in this paper; in [18] we did not have overbars. From Eq. (56) we have

$$-\bar{v}_{,\alpha} + \bar{v}^{(v)}_\alpha \equiv V_\alpha - B_\alpha + AB_\alpha + 2V^\beta C_{\alpha\beta} + \frac{\delta\mu+\delta p}{\mu+p} \times(V_\alpha - B_\alpha) + \frac{\Pi_{\alpha\beta}}{\mu+p}(V^\beta - B^\beta). \quad (111)$$

It is more convenient to introduce the decomposition in Eq. (84). Thus, we have

$$-\bar{v}_{,\alpha} + \bar{v}^{(v)}_\alpha \equiv -v_{,\alpha} + v^{(v)}_\alpha + \frac{\delta\mu+\delta p}{\mu+p}(-v_{,\alpha} + v^{(v)}_\alpha) + \frac{\Pi_{\alpha\beta}}{\mu+p}(-v^{,\beta} + v^{(v)\beta}). \quad (112)$$

The variable $\bar{v}^{(v)}_\alpha$ introduced in [18] cannot be regarded as a proper vector-type perturbation. However, if we also take the temporal comoving gauge in [18] which sets $\bar{v} \equiv 0$ together with $\bar{v}^{(v)}_\alpha = 0$, we have $v = 0 = v^{(v)}_\alpha$; these are the same as taking the proper irrotational condition ($v^{(v)}_\alpha = 0$) and the temporal comoving gauge ($v = 0$). Our analyses in [19,20] are, in fact, based on taking these two conditions together. In the irrotational fluids, the temporal comoving gauge $v \equiv 0$ leads to $\tilde{u}_\alpha = 0$, thus $\tilde{u}_a$ coincides with the normal-frame four-vector $\tilde{n}_a$.

From Eq. (65) we have

$$\tilde{\omega}_{(i)\alpha\beta} = a v^{(v)}_{(i)[\alpha|\beta]} + \frac{a}{\mu_{(i)} + p_{(i)}}(-v_{(i),[\alpha} + v^{(v)}_{(i)[\alpha}) \times \left[\left(\delta p_{(i)} + \frac{2}{3}\frac{\Delta + 3K}{a^2}\Pi_{(i)} - \delta I_{(i)} \right)_{,\beta]} + \frac{\Delta + 3K}{2a}\Pi^{(v)}_{(i)\beta]} - \delta I^{(v)}_{(i)\beta]} \right], \tag{113}$$

where we used Eq. (75) to the linear order. For vanishing vector-type perturbation we set $v^{(v)}_{(i)\alpha} \equiv 0$, etc. In this case, we have

$$\tilde{\omega}_{(i)\alpha\beta} = -\frac{a}{\mu_{(i)} + p_{(i)}} v_{(i),[\alpha} \times \left(\delta p_{(i)} + \frac{2}{3}\frac{\Delta + 3K}{a^2}\Pi_{(i)} - \delta I_{(i)} \right)_{,\beta]}, \tag{114}$$

which vanishes for $\delta p_{(i)} = 0 = \Pi_{(i)}$ and $\delta I_{(i)} = 0$.

VI. EFFECTS OF PRESSURE

We consider a single-component situation without rotational perturbations. Equations (69), (70), (72), and (73) to the linear order provide a complete set of equations we need in the following.

A. Irrotational case

As an irrotational fluid we *ignore* all vector-type perturbations, thus $v^{(v)}_\alpha = B^{(v)}_\alpha = C^{(v)}_\alpha = \Pi^{(v)}_\alpha = 0$. Quadratic combinations of linear-order perturbations of all three types of perturbations contribute to each type of second-order perturbations. Thus, concerning the second-order scalar-type and tensor-type perturbations, ignoring the vector-type perturbation corresponds to ignoring the vector-type perturbation only to the linear order; to the linear order, due to the angular momentum conservation, the vector-type perturbation only has a pure decaying solution in the expanding phase, see Sec. IX. Assuming the background equations are valid, Eqs. (69), (70), (72), and (73) give

$$\dot{\delta} + 3H\left(\frac{\delta p}{\mu} - w\delta\right) - (1+w)\left(\kappa + \frac{\Delta}{a}v - 3H\alpha\right) = -\frac{1}{a}\delta_{,\alpha}\beta^{,\alpha} + \left(\delta + \frac{\delta p}{\mu}\right)(\kappa - 3H\alpha) + (1+w)\left(\kappa + \frac{\Delta}{a}v\right)\alpha + \frac{3}{2}H(1+w)(\alpha^2 - \beta^{,\alpha}\beta_{,\alpha}) - \frac{1}{\mu}\Pi^{\alpha\beta}\left(\frac{1}{a^2}\chi_{,\alpha|\beta} + \dot{C}^{(t)}_{\alpha\beta}\right) - \frac{2}{a}(1+w)v^{,\alpha|\beta}(\varphi g^{(3)}_{\alpha\beta} + \gamma_{,\alpha|\beta} + C^{(t)}_{\alpha\beta}) + \frac{1}{a}(1+w)v^{,\alpha} \times \left\{ a\left(H + \frac{\dot{p}}{\mu + p}\right)v_{,\alpha} - \frac{2}{\mu + p}(\delta p_{,\alpha} + \Pi^\beta_{\alpha|\beta}) + \varphi_{,\alpha} - [(\Delta + 4K)\gamma]_{,\alpha} \right\} + \frac{1}{a\mu}[(\delta\mu + \delta p)v^{,\alpha} + \Pi^{\alpha\beta}v_{,\beta}]_{|\alpha}, \tag{115}$$

$$\frac{1}{a^4}[a^4(\mu + p)v_{,\alpha}]^{\cdot} - \frac{1}{a}[(\mu + p)\alpha_{,\alpha} + \delta p_{,\alpha} + \Pi^\beta_{\alpha|\beta}] = \frac{1}{a}(\mu + p)(v - \beta)^{,\beta}(v - \beta)_{,\alpha|\beta} + (\mu + p)v_{,\alpha}\left(\kappa + \frac{\Delta}{a}v - 3H\alpha\right) - \frac{1}{a^4}\{a^4[(\delta\mu + \delta p)v_{,\alpha} + \Pi^\beta_\alpha v_{,\beta}]\}^{\cdot} - \frac{1}{a}\{-(\delta\mu + \delta p)\alpha_{,\alpha} + \alpha[(\mu + p)\alpha_{,\alpha} - \delta p_{,\alpha} - \Pi^\beta_{\alpha|\beta}] - \alpha_{,\beta}\Pi^\beta_\alpha + 2(C^{\gamma\beta}\Pi_{\alpha\gamma})_{|\beta} - \Pi^\gamma_\alpha C^\beta_{\beta|\gamma} + \Pi^\beta_\gamma C^\gamma_{\beta|\alpha}\}, \tag{116}$$

$$\dot{\kappa} + 2H\kappa - 4\pi G(\delta\mu + 3\delta p) + \left(3\dot{H} + \frac{\Delta}{a^2}\right)\alpha = -\frac{1}{a}\kappa_{,\alpha}\beta^{,\alpha} + \alpha\dot{\kappa} + \frac{1}{3}\kappa^2 + \frac{3}{2}\dot{H}(3\alpha^2 - \beta^{,\alpha}\beta_{,\alpha}) + 8\pi G(\mu + p)v^{,\alpha}v_{,\alpha} + \frac{1}{a^2}\left\{2\Delta\alpha(\alpha + \varphi) + \alpha^{,\alpha}[\alpha - \varphi + (\Delta + 4K)\gamma]_{,\alpha} + 2\alpha^{,\alpha|\beta}(\gamma_{,\alpha|\beta} + C^{(t)}_{\alpha\beta}) - \frac{1}{2}\Delta(\beta^{,\alpha}\beta_{,\alpha})\right\} + \left(\frac{1}{a^2}\chi^{,\alpha|\beta} + \dot{C}^{(t)\alpha\beta}\right)\left(\frac{1}{a^2}\chi_{,\alpha|\beta} + \dot{C}^{(t)}_{\alpha\beta}\right) - \frac{1}{3}\left(\frac{\Delta}{a^2}\chi\right)^2, \tag{117}$$

$$\kappa = -\frac{\Delta + 3K}{a^2}\chi + 12\pi Ga(\mu + p)v. \tag{118}$$

Equation (118) is valid to the linear order.

B. Comoving gauge

As the temporal and spatial gauge conditions we set

$$v \equiv 0 \equiv \gamma, \tag{119}$$

thus, $\beta = \chi/a$. Under these gauge conditions Eq. (116) gives

$$\alpha = -\frac{1}{2a^2}\chi^{\alpha}\chi_{,\alpha} - \frac{c_s^2}{1+w}\delta\left[1 - \frac{1+3c_s^2}{2(1+w)}\delta\right] + \alpha_{\Pi}, \tag{120}$$

where the imperfect fluid contributions (stresses e and $\Pi_{\alpha\beta}$) are denoted by α_{Π} with

$$\alpha_{\Pi} \equiv \frac{1}{\mu+p}\left(e + \frac{2}{3}\frac{\Delta+3K}{a^2}\Pi\right)\left[-1 + \frac{2c_s^2}{1+w}\delta + \frac{1}{\mu+p}\left(e + \frac{2}{3}\frac{\Delta+3K}{a^2}\Pi\right)\right] + \frac{1}{\mu+p}\Delta^{-1}\nabla^{\alpha}\left[\frac{1+c_s^2}{1+w}\delta(e_{,\alpha} + \Pi^{\beta}_{\alpha|\beta}) - \left(\frac{c_s^2}{1+w}\delta_{,\beta} + \frac{e_{,\beta} + \Pi^{\gamma}_{\beta|\gamma}}{\mu+p}\right)\Pi^{\beta}_{\alpha} + 2(C^{\beta\gamma}\Pi_{\alpha\gamma})_{|\beta} - \Pi^{\gamma}_{\alpha}C^{\beta}_{\beta|\gamma} + \Pi^{\beta}_{\gamma}C^{\gamma}_{\beta|\alpha}\right]. \tag{121}$$

Using this, Eqs. (115) and (118) give

$$\dot{\delta} - 3wH\delta - (1+w)\kappa = -\frac{1}{a^2}\delta_{,\alpha}\chi^{\alpha} + \kappa\delta + \frac{3}{2}H\frac{c_s^2}{1+w}\delta^2 + \delta_{\Pi}, \tag{122}$$

$$\dot{\kappa} + 2H\kappa - 4\pi G\mu\delta - \frac{c_s^2}{1+w}\frac{\Delta+3K}{a^2}\delta = -\frac{1}{a^2}\kappa_{,\alpha}\chi^{\alpha} + \frac{1}{3}\kappa^2 + \frac{c_s^2}{2(1+w)}\left(4\pi G\mu - \frac{1+2c_s^2}{1+w}\frac{\Delta+3K}{a^2}\right)\delta^2 + \frac{c_s^2}{1+w}\left[2H\delta\kappa + \frac{1}{a^2}(\varphi^{,\alpha}\delta_{,\alpha} - 2\varphi\Delta\delta - 2\delta^{,\alpha|\beta}C^{(t)}_{\alpha\beta})\right] + \left(\frac{1}{a^2}\chi^{\alpha|\beta} + \dot{C}^{(t)\alpha\beta}\right)\left(\frac{1}{a^2}\chi_{,\alpha|\beta} + \dot{C}^{(t)}_{\alpha\beta}\right) - \frac{1}{3}\left(\frac{\Delta}{a^2}\chi\right)^2 + \kappa_{\Pi}, \tag{123}$$

$$\kappa = -\frac{\Delta+3K}{a^2}\chi, \tag{124}$$

where Eq. (124) is valid only to the linear order, and

$$\delta_{\Pi} \equiv 2H\frac{\Delta+3K}{a^2}\frac{\Pi}{\mu} + \frac{e}{\mu}\left(\kappa + 3H\frac{c_s^2}{1+w}\delta\right) - \frac{\Pi^{\alpha\beta}}{\mu}\left(\frac{1}{a^2}\chi_{,\alpha|\beta} + \dot{C}^{(t)}_{\alpha\beta}\right) + \frac{1}{\mu+p}\left(e + \frac{2}{3}\frac{\Delta+3K}{a^2}\Pi\right)\left[3H\delta - (1+w)\kappa + \frac{3}{2}H\frac{1}{\mu}\left(e - \frac{2}{3}\frac{\Delta+3K}{a^2}\Pi\right)\right] - 3H\frac{1}{\mu}\Delta^{-1}\nabla^{\alpha}\left[\frac{1+c_s^2}{1+w}\delta(e_{,\alpha} + \Pi^{\beta}_{\alpha|\beta}) - \left(\frac{c_s^2}{1+w}\delta_{,\beta} + \frac{e_{,\beta} + \Pi^{\gamma}_{\beta|\gamma}}{\mu+p}\right)\Pi^{\beta}_{\alpha} + 2(C^{\beta\gamma}\Pi_{\alpha\gamma})_{|\beta} - \Pi^{\gamma}_{\alpha}C^{\beta}_{\beta|\gamma} + \Pi^{\beta}_{\gamma}C^{\gamma}_{\beta|\alpha}\right],$$

$$\kappa_{\Pi} \equiv 12\pi Ge\left(1 - \frac{c_s^2}{1+w}\delta\right) - \left(3\dot{H} + \frac{\Delta}{a^2}\right)\left\{\frac{1}{\mu+p}\left(e + \frac{2}{3}\frac{\Delta+3K}{a^2}\Pi\right)\left[-1 + \frac{2c_s^2}{1+w}\delta + \frac{1}{\mu+p}\left(e + \frac{2}{3}\frac{\Delta+3K}{a^2}\Pi\right)\right] + \frac{1}{\mu+p}\Delta^{-1}\nabla^{\alpha}\left[\frac{1+c_s^2}{1+w}\delta(e_{,\alpha} + \Pi^{\beta}_{\alpha|\beta}) - \left(\frac{c_s^2}{1+w}\delta_{,\beta} + \frac{e_{,\beta} + \Pi^{\gamma}_{\beta|\gamma}}{\mu+p}\right)\Pi^{\beta}_{\alpha} + 2(C^{\beta\gamma}\Pi_{\alpha\gamma})_{|\beta} - \Pi^{\gamma}_{\alpha}C^{\beta}_{\beta|\gamma} + \Pi^{\beta}_{\gamma}C^{\gamma}_{\beta|\alpha}\right]\right\} - \left[-2H\kappa + 4\pi G(1+6c_s^2)\mu\delta + 12\pi Ge - \frac{c_s^2}{1+w}\frac{3K}{a^2}\delta - \frac{3}{2}\dot{H}\frac{1}{\mu+p}\left(e + \frac{2}{3}\frac{\Delta+3K}{a^2}\Pi\right) + 2\varphi\frac{\Delta}{a^2}\right] \times \frac{1}{\mu+p}\left(e + \frac{2}{3}\frac{\Delta+3K}{a^2}\Pi\right) + \frac{1}{\mu+p}\frac{\Delta}{a^2}\left\{\left(e + \frac{2}{3}\frac{\Delta+3K}{a^2}\Pi\right)\left[\frac{1}{2}\frac{1}{\mu+p}\left(e + \frac{2}{3}\frac{\Delta+3K}{a^2}\Pi\right) + \frac{c_s^2}{1+w}\delta\right]\right\} + \frac{1}{\mu+p}\frac{1}{a^2}\left[\left(e + \frac{2}{3}\frac{\Delta+3K}{a^2}\Pi\right)^{,\alpha}\varphi_{,\alpha} - 2\left(e + \frac{2}{3}\frac{\Delta+3K}{a^2}\Pi\right)^{,\alpha|\beta}C^{(t)}_{\alpha\beta}\right]. \tag{125}$$

Combining Eqs. (122) and (123) we can derive

$$\frac{1+w}{a^2H}\left[\frac{H^2}{(\mu+p)a}\left(\frac{a^3\mu}{H}\delta\right)^{\cdot}\right]^{\cdot} - c_s^2\frac{\Delta}{a^2}\delta = (1+w)\left\{\frac{1}{a^4}\chi^{,\alpha|\beta}\chi_{,\alpha|\beta} - \frac{1}{a^2}\kappa_{,\alpha}\chi^{,\alpha} + \frac{1}{3}\left[\kappa^2 - \left(\frac{\Delta}{a}\chi\right)^2\right]\right\} + \frac{1+w}{a^2}\left[\frac{1}{1+w}\left(a^2\kappa\delta - \delta_{,\alpha}\chi^{,\alpha} + \frac{3}{2}a^2H\frac{c_s^2}{1+w}\delta^2\right)\right]^{\cdot} + \frac{1}{2}c_s^2\left(4\pi G\mu - \frac{1+2c_s^2}{1+w}\frac{\Delta+3K}{a^2}\right)\delta^2 + c_s^2\frac{1}{a^2}(2a^2H\delta\kappa + \varphi^{,\alpha}\delta_{,\alpha} - 2\varphi\Delta\delta - 2\delta^{,\alpha|\beta}C^{(t)}_{\alpha\beta}) + (1+w)\dot{C}^{(t)\alpha\beta}\left(\frac{2}{a^2}\chi_{,\alpha|\beta} + \dot{C}^{(t)}_{\alpha\beta}\right) + (1+w)\kappa_\Pi + \frac{1+w}{a^2}\left(\frac{a^2}{1+w}\delta_\Pi\right)^{\cdot} - \frac{1}{a^2}\delta_{\Pi,\alpha}\chi^{,\alpha}. \quad (126)$$

Notice that the equations above are valid in the presence of general K and Λ.

C. Newtonian correspondence

Guided by our success in the zero-pressure case, we continue to *identify*

$$\kappa \equiv -\frac{1}{a}\nabla\cdot\mathbf{u}, \quad (127)$$

to the second order. Using Eq. (124), *assuming* $K = 0$, we can identify

$$\kappa = -c\frac{\Delta}{a^2}\chi = -\frac{1}{a}\nabla\cdot\mathbf{u}, \qquad \mathbf{u} \equiv \frac{c}{a}\nabla\chi = -c\nabla v_\chi, \quad (128)$$

to the linear order. We have recovered the speed of light c. We have $[\mathbf{u}] = LT^{-1}$. Equation (120) becomes

$$\alpha = -\frac{1}{2c^2}\mathbf{u}^2 - \frac{c_s^2}{1+w}\delta\left[1 - \frac{1+3c_s^2}{2(1+w)}\delta\right] + \alpha_\Pi. \quad (129)$$

Equations (122) and (123) give

$$\dot\delta - 3wH\delta + (1+w)\frac{1}{a}\nabla\cdot\mathbf{u} = -\frac{1}{a}\nabla\cdot(\delta\mathbf{u}) + \frac{3}{2}\frac{c_s^2}{1+w}H\delta^2 + \delta_\Pi, \quad (130)$$

$$\frac{1}{a}\nabla\cdot(\dot{\mathbf{u}} + H\mathbf{u}) + \frac{4\pi G\mu}{c^2}\delta + \frac{c_s^2}{1+w}c^2\frac{\Delta}{a^2}\delta = -\frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) - \dot{C}^{(t)\alpha\beta}\left(\frac{2}{a}u_{\alpha|\beta} + \dot{C}^{(t)}_{\alpha\beta}\right) + \frac{c_s^2}{1+w}\times\left\{\frac{1}{2}\left(-\frac{4\pi G\mu}{c^2} + \frac{1+2c_s^2}{1+w}c^2\frac{\Delta}{a^2}\right)\delta^2 + 2H\delta\frac{1}{a}\nabla\cdot\mathbf{u} + \frac{c^2}{a^2}[2\varphi\Delta\delta - (\nabla\varphi)\cdot\nabla\delta + 2\delta^{,\alpha|\beta}C^{(t)}_{\alpha\beta}]\right\} - \kappa_\Pi. \quad (131)$$

We have $[\alpha_\Pi] = 1$, $[\delta_\Pi] = T^{-1}$, and $[\kappa_\Pi] = T^{-2}$. Combining these equations or Eq. (126) gives

$$\frac{1+w}{a^2H}\left[\frac{H^2}{(\mu+p)a}\left(\frac{a^3\mu}{H}\delta\right)^{\cdot}\right]^{\cdot} - c_s^2c^2\frac{\Delta}{a^2}\delta = \frac{1+w}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) - \frac{1+w}{a^2}\left\{\frac{a}{1+w}\left[\nabla\cdot(\delta\mathbf{u}) - \frac{3}{2}aH\frac{c_s^2}{1+w}\delta^2\right]\right\}^{\cdot} + (1+w)\dot{C}^{(t)\alpha\beta}\left(\frac{2}{a}u_{\alpha|\beta} + \dot{C}^{(t)}_{\alpha\beta}\right) + \frac{1}{2}c_s^2\left(\frac{4\pi G\mu}{c^2} - \frac{1+2c_s^2}{1+w}c^2\frac{\Delta}{a^2}\right)\delta^2 - c_s^2\frac{1}{a^2}[2aH\delta\nabla\cdot\mathbf{u} + 2\varphi c^2\Delta\delta - c^2(\nabla\varphi)\cdot\nabla\delta + 2c^2\delta^{,\alpha|\beta}C^{(t)}_{\alpha\beta}] + (1+w)\kappa_\Pi + \frac{1+w}{a^2}\left(\frac{a^2}{1+w}\delta_\Pi\right)^{\cdot} - \frac{1}{a}\mathbf{u}\cdot\nabla\delta_\Pi. \quad (132)$$

We note that, to the linear order, Eqs. (130)–(132) are valid in the presence of general K and Λ; compare with Eqs. (122), (123), and (126). To the second order Eqs. (130) and (132) are valid for $K = 0$, but in the presence of general Λ.

D. Linear-order relativistic pressure corrections

To the linear order, *ignoring* the entropic perturbation e and the anisotropic stress Π, Eqs. (130)–(132) become

$$\dot\delta - 3wH\delta + (1+w)\frac{1}{a}\nabla\cdot\mathbf{u} = 0, \quad (133)$$

$$\frac{1}{a}\nabla\cdot(\dot{\mathbf{u}} + H\mathbf{u}) + 4\pi G\varrho\delta = -\frac{1}{1+w}\frac{\Delta}{a^2}\frac{\delta p}{\varrho}, \quad (134)$$

$$\frac{1+w}{a^2H}\left[\frac{H^2}{a(1+w)\varrho}\left(\frac{a^3\varrho}{H}\delta\right)^{\cdot}\right]^{\cdot} = \frac{\Delta}{a^2}\frac{\delta p}{\varrho}, \quad (135)$$

where *ignoring* the specific internal energy density ϵ we used $\mu = \varrho c^2$, thus $\delta = \delta\varrho/\varrho$; in general we have $\mu = \varrho(c^2 + \epsilon)$ [22]. These equations are valid for general K and Λ. Equation (135) can be expanded to give

$$\frac{1+w}{a^2 H}\left[\frac{H^2}{a(1+w)\varrho}\left(\frac{a^3\varrho}{H}\delta\right)^{\cdot}\right]^{\cdot} = \frac{\Delta}{a^2}\frac{\delta p}{\varrho} = \ddot{\delta} + (2 - 6w + 3c_s^2)H\dot{\delta} - \left[(1 + 8w - 6c_s^2 - 3w^2)4\pi G\varrho - 12(w - c_s^2)\frac{Kc^2}{a^2} + (5w - 3c_s^2)\Lambda c^2\right]\delta, \quad (136)$$

which is valid in the presence of K and Λ [21]. Equation of linear density perturbation in the comoving gauge was first derived by Nariai in [11].

In a single-component case the Newtonian equations in Eqs. (8), (12), and (16) give

$$\dot{\delta} + \frac{1}{a}\nabla\cdot\mathbf{u} = 0, \quad (137)$$

$$\frac{1}{a}\nabla\cdot(\dot{\mathbf{u}} + H\mathbf{u}) + 4\pi G\varrho\delta = -\frac{\Delta}{a^2}\frac{\delta p}{\varrho}, \quad (138)$$

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\varrho\delta = \frac{1}{a^2 H}\left[a^2 H^2\left(\frac{1}{H}\delta\right)^{\cdot}\right]^{\cdot} = \frac{\Delta}{a^2}\frac{\delta p}{\varrho}. \quad (139)$$

Comparing Eqs. (133) and (134) with Eqs. (137) and (138), we notice the presence of the $w \equiv p/(\varrho c^2)$ term in three places in the relativistic equations; for $w = 0$ we recover the Newtonian equations. Even to the linear order the presence of these terms should be regarded as a pure general relativistic effect of the isotropic pressure. The effects of pressures to the second order compared with the Newtonian equations can be found in Eqs. (130)–(132) which should be compared with Newtonian equations in Eqs. (8), (12), and (16). Pressure has the genuine relativistic role in cosmology even in the background level; compare Eqs. (76)–(79) with Eqs. (7).

We note again that, in the relativistic situation to the second order, the density perturbation δ and the perturbed velocity $\mathbf{u}$ are defined as

$$\delta \equiv \delta_v, \qquad \kappa_v \equiv -\frac{1}{a}\nabla\cdot\mathbf{u}, \quad (140)$$

where δ_v and κ_v are the relative density perturbation $\delta\mu/\mu$ and the perturbed trace of extrinsic curvature (or perturbed expansion scalar of the normal frame with negative sign), respectively, both in the temporal comoving

TABLE I. Symbols used in Sec. VI.

Symbol	Definition	Equation
a	Cosmic scale factor	(5) and (50)
H	Hubble parameter ($\equiv \dot{a}/a$)	(4)
K	Sign of spatial curvature	(79)
Λ	The cosmological constant	(79)
μ	Background energy density	(52)
p	Background pressure	(52)
w	Equation of state parameter ($\equiv p/\mu$)	(88)
c_s^2	Sound velocity parameter ($\equiv \dot{p}/\dot{\mu}$)	(88)
$\delta\mu$	Perturbed energy density	(52)
δ	Relative density perturbations ($\equiv \delta\mu/\mu$)	
δp	Perturbed pressure	(52)
e	Entropic perturbation ($\equiv \delta p - c_s^2\delta\mu$)	(88)
$\Pi_{\alpha\beta}$	Anisotropic stress	(52)
Π	Scalar part of the anisotropic stress	(52) and (85)
α	Metric perturbation variable	(50) and (81)
γ	Metric perturbation variable; as a spatial gauge condition we set $\gamma \equiv 0$	(50) and (81)
β	Metric perturbation variable	(50) and (81)
φ	Metric perturbation variable, perturbed spatial curvature	(50) and (81)
χ	Metric perturbation variable; temporal zero-shear gauge sets $\chi \equiv 0$	(82)
κ	Perturbed part of trace of extrinsic curvature; negative of perturbed expansion	(83)
v	Scalar part of v_α; temporal comoving gauge sets $v \equiv 0$	(84)
v_χ	A gauge-invariant combination using v and χ which becomes v in the zero-shear gauge ($\chi \equiv 0$)	
χ_v	A gauge-invariant combination using χ and v which becomes χ in the comoving gauge ($v \equiv 0$)	
$\mathbf{u}$	Relativistic velocity perturbation of collective component identified from κ_v	(127)
$C_{\alpha\beta}$	Spatial part of metric perturbation	(50)
$C^{(t)}_{\alpha\beta}$	Transverse-tracefree part of metric perturbation variable	(50) and (81)

gauge ($v \equiv 0$) and spatial ($\gamma = 0$) gauge; these are equivalently gauge-invariant combinations which become δ and κ in the comoving gauge condition. δp is also a gauge-invariant combination δp_v which becomes δp in the comoving gauge.

In Table I we summarize symbols used in this section.

VII. EFFECTS OF MULTICOMPONENT

We *assume* zero-pressure medium, thus set

$$p_{(i)} \equiv 0, \qquad \delta p_{(i)} \equiv 0 \equiv \Pi_{(i)\alpha\beta}. \tag{141}$$

From Eq. (93) we notice that to the second order the collective fluid quantities differ from a simple sum of individual fluid quantities. Even in the zero-pressure mediums, we have

$$\begin{aligned} \delta\mu &= \sum_j [\delta\mu_{(j)} + \mu_{(j)} v^\alpha_{(j)}(v_{(j)\alpha} - v_\alpha)], \\ \delta p &= \tfrac{1}{3}\sum_j \mu_{(j)} v^\alpha_{(j)}(v_{(j)\alpha} - v_\alpha), \\ \mu v_\alpha &= \sum_j [\mu_{(j)} v_{(j)\alpha} + \delta\mu_{(j)}(v_{(j)\alpha} - v_\alpha)], \\ \Pi^\alpha_\beta &= \sum_j \mu_{(j)}[v^\alpha_{(j)}(v_{(j)\beta} - v_\beta) - \tfrac{1}{3}\delta^\alpha_\beta v^\gamma_{(j)}(v_{(j)\gamma} - v_\gamma)], \end{aligned} \tag{142}$$

thus $\delta p \neq 0 \neq \Pi^\alpha_\beta$ to the second order.

Equations (69), (70), and (72)–(75) give

$$\begin{aligned} &\dot\delta_{(i)} - \delta K + 3HA + \frac{1}{a}[(1+\delta_{(i)})v^\alpha_{(i)}]_{|\alpha} \\ &\quad = -\frac{1}{a}\delta_{(i),\alpha}B^\alpha + \delta_{(i)}(\delta K - 3HA) + A\delta K \\ &\qquad + \frac{3}{2}H(A^2 - B^\alpha B_\alpha) - v_{(i)\alpha}(2\dot v^\alpha_{(i)} + Hv^\alpha_{(i)}) \\ &\qquad - \frac{1}{a}Av^\alpha_{(i)|\alpha} - \frac{2}{a}A_{,\alpha}v^\alpha_{(i)} + \frac{2}{a}(C^{\alpha\beta}v_{(i)\beta})_{|\alpha} \\ &\qquad - \frac{1}{a}C^{\alpha|\beta}_\alpha v_{(i)\beta}, \end{aligned} \tag{143}$$

$$\begin{aligned} \frac{1}{a}[a(1+\delta_{(i)})v_{(i)\alpha}]^{\cdot} + \frac{1}{a}A_{,\alpha} &= (\delta K - 3HA)v_{(i)\alpha} \\ &\quad + \frac{1}{a}[AA_{,\alpha} - \delta_{(i)}A_{,\alpha} \\ &\quad - B^\beta B_{\beta|\alpha} - (v_{(i)\alpha}v^\beta_{(i)})_{|\beta} \\ &\quad - v_{(i)\alpha|\beta}B^\beta - v_{(i)\beta}B^\beta{}_{|\alpha}], \end{aligned} \tag{144}$$

$$\begin{aligned} &\delta\dot K + 2H\delta K - 4\pi G(\delta\mu + 3\delta p) + \left(3\dot H + \frac{\Delta}{a^2}\right)A \\ &\quad = A\delta\dot K - \frac{1}{a}\delta K_{,\alpha}B^\alpha + \frac{1}{3}\delta K^2 + \frac{3}{2}\dot H(3A^2 - B^\alpha B_\alpha) \\ &\qquad + \frac{1}{a^2}\Big[2A\Delta A + A^{,\alpha}A_{,\alpha} - \frac{1}{2}\Delta(B^\alpha B_\alpha) \\ &\qquad + A^{,\alpha}(2C^\beta_{\alpha|\beta} - C^\beta_{\beta|\alpha}) + 2C^{\alpha\beta}A_{,\alpha|\beta}\Big] \\ &\qquad + \left(\dot C^{\alpha\beta} + \frac{1}{a}B^{(\alpha|\beta)}\right)\left(\dot C_{\alpha\beta} + \frac{1}{a}B_{\alpha|\beta}\right) \\ &\qquad - \frac{1}{3}\left(\dot C^\alpha_\alpha + \frac{1}{a}B^\alpha{}_{|\alpha}\right)^2 + 8\pi G\mu v^\alpha v_\alpha, \end{aligned} \tag{145}$$

$$\begin{aligned} &\dot\delta - \delta K + 3HA + \frac{1}{a}[(1+\delta)v^\alpha]_{|\alpha} \\ &\quad = -\frac{1}{a}\delta_{,\alpha}B^\alpha + \delta(\delta K - 3HA) + A\delta K \\ &\qquad + \frac{3}{2}H(A^2 - B^\alpha B_\alpha) - v_\alpha(2\dot v^\alpha + Hv^\alpha) \\ &\qquad - \frac{1}{a}Av^\alpha{}_{|\alpha} - \frac{2}{a}A_{,\alpha}v^\alpha + \frac{2}{a}(C^{\alpha\beta}v_\beta)_{|\alpha} \\ &\qquad - \frac{1}{a}C^{\alpha|\beta}_\alpha v_\beta - 3H\frac{\delta p}{\mu}, \end{aligned} \tag{146}$$

$$\begin{aligned} \frac{1}{a}[a(1+\delta)v_\alpha]^{\cdot} + \frac{1}{a}A_{,\alpha} &= (\delta K - 3HA)v_\alpha \\ &\quad + \frac{1}{a}[AA_{,\alpha} - \delta_{(i)}A_{,\alpha} - B^\beta B_{\beta|\alpha} \\ &\quad - (v_\alpha v^\beta)_{|\beta} - v_{\alpha|\beta}B^\beta - v_\beta B^\beta{}_{|\alpha} \\ &\quad - \frac{1}{\mu}(\delta p_{,\alpha} + \Pi^\beta_{\alpha|\beta})], \end{aligned} \tag{147}$$

$$\begin{aligned} &\left[\dot C^\beta_\alpha + \frac{1}{2a}\left(B^\beta{}_{|\alpha} + B_\alpha{}^{|\beta}\right)\right]_{|\beta} - \frac{1}{3}\left(\dot C^\gamma_\gamma + \frac{1}{a}B^\gamma{}_{|\gamma}\right)_{,\alpha} \\ &\qquad + \frac{2}{3}\delta K_{,\alpha} = -8\pi Ga\sum_j \mu_{(j)}v_{(j)\alpha}. \end{aligned} \tag{148}$$

where Eq. (148) is valid to the linear order. Notice that we have not taken any gauge condition in the above equations.

A. Irrotational case

Assuming an irrotational condition we ignore all vector-type perturbations. As the spatial gauge condition we take

$$\gamma \equiv 0, \tag{149}$$

thus, $\beta \equiv \chi/a$. Equations (143)–(148) become

$$\dot{\delta}_{(i)} - \kappa + 3H\alpha - \frac{1}{a}[(1+\delta_{(i)})v^{,\alpha}_{(i)}]_{|\alpha} = -\frac{1}{a^2}\delta_{(i),\alpha}\chi^{,\alpha} + \delta_{(i)}(\kappa - 3H\alpha) + \alpha\kappa + \frac{3}{2}H\left(\alpha^2 - \frac{1}{a^2}\chi^{,\alpha}\chi_{,\alpha}\right) + Hv_{(i),\alpha}v^{,\alpha}_{(i)} + \frac{1}{a}\alpha\Delta v_{(i)} + \frac{1}{a}\varphi^{,\alpha}v_{(i),\alpha} - \frac{2}{a}\varphi\Delta v_{(i)} - \frac{2}{a}C^{(t)\alpha\beta}v_{(i),\alpha|\beta}, \tag{150}$$

$$\frac{1}{a}[a(1+\delta_{(i)})v_{(i)}]^{\cdot}_{,\alpha} - \frac{1}{a}\alpha_{,\alpha} = (\kappa - 3H\alpha)v_{(i),\alpha} + \frac{1}{a}\times\left[-\alpha\alpha_{,\alpha} + \frac{1}{a^2}\chi^{,\beta}\chi_{,\beta|\alpha} + \delta_{(i)}\alpha_{,\alpha} + (v_{(i),\alpha}v^{,\beta}_{(i)})_{|\beta} - \frac{1}{a}(v_{(i),\beta}\chi^{,\beta})_{,\alpha}\right], \tag{151}$$

$$\dot{\kappa} + 2H\kappa - 4\pi G(\delta\mu + 3\delta p) + \left(3\dot{H} + \frac{\Delta}{a^2}\right)\alpha = \alpha\dot{\kappa} - \frac{1}{a^2}\kappa_{,\alpha}\chi^{,\alpha} + \frac{1}{3}\kappa^2 + \frac{3}{2}\dot{H}\left(3\alpha^2 - \frac{1}{a^2}\chi^{,\alpha}\chi_{,\alpha}\right) + \frac{1}{a^2}\left[2\alpha\Delta\alpha + \alpha^{,\alpha}\alpha_{,\alpha} - \frac{\Delta}{2a^2}(\chi^{,\alpha}\chi_{,\alpha}) - \alpha^{,\alpha}\varphi_{,\alpha} + 2\varphi\Delta\alpha + 2C^{(t)\alpha\beta}\alpha_{,\alpha|\beta}\right] + \left(\dot{C}^{(t)\alpha\beta} + \frac{1}{a^2}\chi^{,\alpha|\beta}\right)\times\left(\dot{C}^{(t)}_{\alpha\beta} + \frac{1}{a^2}\chi_{,\alpha|\beta}\right) - \frac{1}{3}\left(\frac{\Delta}{a^2}\chi\right)^2 + 8\pi G\mu v^{,\alpha}v_{,\alpha}, \tag{152}$$

$$\dot{\delta} - \kappa + 3H\alpha - \frac{1}{a}[(1+\delta)v^{,\alpha}]_{|\alpha} = -\frac{1}{a^2}\delta_{,\alpha}\chi^{,\alpha} + \delta(\kappa - 3H\alpha) + \alpha\kappa + \frac{3}{2}H\left(\alpha^2 - \frac{1}{a^2}\chi^{,\alpha}\chi_{,\alpha}\right) + Hv_{,\alpha}v^{,\alpha} + \frac{1}{a}\alpha\Delta v + \frac{1}{a}\varphi^{,\alpha}v_{,\alpha} - \frac{2}{a}\varphi\Delta v - \frac{2}{a}C^{(t)\alpha\beta}v_{,\alpha|\beta} - 3H\frac{\delta p}{\mu}, \tag{153}$$

$$\frac{1}{a}[a(1+\delta)v]^{\cdot}_{,\alpha} - \frac{1}{a}\alpha_{,\alpha} = (\kappa - 3H\alpha)v_{,\alpha} + \frac{1}{a}\left[-\alpha\alpha_{,\alpha} + \frac{1}{a^2}\chi^{,\beta}\chi_{,\beta|\alpha} + \delta\alpha_{,\alpha} + (v_{,\alpha}v^{,\beta})_{|\beta} - \frac{1}{a}(v_{,\beta}\chi^{,\beta})_{,\alpha} - \frac{1}{\mu}(\delta p_{,\alpha} + \Pi^{\beta}_{\alpha|\beta})\right], \tag{154}$$

$$\frac{\Delta + 3K}{a^2}\chi + \kappa = 12\pi Ga\sum_j \mu_{(j)}v_{(j)}. \tag{155}$$

Equation (155) is valid to the linear order. Notice that we have not taken the temporal gauge condition in the above equations.

B. Linear perturbations

To the linear order Eqs. (150)–(154) give

$$\dot{\delta}_{(i)} - \kappa + 3H\alpha - c\frac{\Delta}{a}v_{(i)} = 0, \tag{156}$$

$$\dot{v}_{(i)} + Hv_{(i)} - \frac{c}{a}\alpha = 0, \tag{157}$$

$$\dot{\kappa} + 2H\kappa - 4\pi G\sum_j \varrho_{(j)}\delta_{(j)} + \left(3\dot{H} + c^2\frac{\Delta}{a^2}\right)\alpha = 0, \tag{158}$$

$$c\frac{\Delta + 3K}{a^2}\chi + \kappa = \frac{12\pi G}{c}a\sum_j \varrho_{(j)}v_{(j)}, \tag{159}$$

$$\dot{\delta} - \kappa + 3H\alpha - c\frac{\Delta}{a}v = 0, \tag{160}$$

$$\dot{v} + Hv - \frac{c}{a}\alpha = 0, \tag{161}$$

where we have recovered the speed of light c. The following additional equations follow from Eqs. (95), (96), and (99)

$$\kappa - 3H\alpha + 3\dot{\varphi} + c\frac{\Delta}{a^2}\chi = 0, \tag{162}$$

$$4\pi G\varrho\delta + H\kappa + c^2\frac{\Delta + 3K}{a^2}\varphi = 0, \tag{163}$$

$$\frac{1}{c}(\dot{\chi} + H\chi) - \varphi - \alpha = 0. \tag{164}$$

Equation (164) gives

$$\alpha_\chi = -\varphi_\chi. \tag{165}$$

Equation (161) gives

$$\alpha_v = 0. \tag{166}$$

Equations (159) and (162) give

$$\dot{\varphi}_v = \frac{Kc}{a^2}\chi_v. \tag{167}$$

Thus, for $K = 0$ we have

$$\dot{\varphi}_v = 0, \tag{168}$$

which is valid even in the presence of multicomponents and the cosmological constant. Equations (157) and (161) give

$$\dot{v}_\chi + Hv_\chi + \frac{c}{a}\varphi_\chi = 0, \tag{169}$$

$$\dot{v}_{(i)\chi} + Hv_{(i)\chi} + \frac{c}{a}\varphi_\chi = 0. \tag{170}$$

Equations (159) and (163) give

$$c^2\frac{\Delta + 3K}{a^2}\varphi_\chi = -4\pi G\varrho\delta_v = -4\pi G\sum_j \varrho_{(j)}\delta_{(j)v} = -4\pi G\sum_j \varrho_{(j)}\delta_{(j)v_{(j)}}. \tag{171}$$

We can derive a density perturbation equation in many different temporal gauge (hypersurface) conditions all of which naturally correspond to gauge-invariant variables. In a single-component case, density perturbation in the comoving gauge ($v = 0$) is known to give a Newtonian result. In the multicomponent situation we have many different comoving gauge conditions. Here, we consider two such gauges for the $\delta_{(i)}$ variable: one based on the $v = 0$ gauge, and the other based on the $v_{(\ell)} = 0$ gauge for a specific ℓ.

(i) Equation (156) evaluated in the $v = 0$ gauge, and using Eq. (159) we can derive

$$\dot{\delta}_{(i)v} - c\frac{\Delta}{a}v_{(i)\chi} - c\frac{3K}{a}v_\chi = 0, \tag{172}$$

where we used $\chi_v \equiv \chi - av \equiv -av_\chi$, $v_{v_{(i)}} \equiv v - v_{(i)}$, and $\alpha_v \equiv \alpha - c^{-1}(av)^\cdot = 0$. Using Eqs. (169)–(171) we have

$$\ddot{\delta}_{(i)v} + 2H\dot{\delta}_{(i)v} - 4\pi G\sum_j \varrho_{(j)}\delta_{(j)v} = 0. \tag{173}$$

This coincides exactly with the Newtonian result in Eq. (16) to the linear order, even in the presence of K. Thus, we may identify $\delta_{(i)v}$ as the Newtonian density perturbation δ_i to the linear order even in the presence of K. For $K = 0$ we may also identify $-c\nabla v_{(i)\chi}$ as the Newtonian velocity perturbation $\mathbf{u}_i$. However, in the presence of K, we cannot identify the relativistic variables which correspond to the Newtonian velocity perturbation of the individual component. Therefore, to the linear order we have the following Newtonian correspondences

$$\delta_i = \delta_{(i)v}, \qquad \mathbf{u}_i = -c\nabla v_{(i)\chi}, \tag{174}$$

where the latter one is valid only for $K = 0$.

(ii) Evaluating Eq. (156) in the $v_{(\ell)} = 0$ gauge for a specific ℓ, and using Eq. (159) we can derive

$$\dot{\delta}_{(i)v_{(\ell)}} - c\frac{\Delta + 3K}{a}v_{(\ell)\chi} = \frac{12\pi G}{c}a\sum_j \varrho_{(j)}(v_{(j)} - v_{(\ell)}) + c\frac{\Delta}{a}(v_{(i)} - v_{(\ell)}), \tag{175}$$

where we used $\chi_{v_{(\ell)}} \equiv \chi - av_{(\ell)} \equiv -av_{(\ell)\chi}$, $v_{(i)v_{(\ell)}} \equiv v_{(i)} - v_{(\ell)}$, and $\alpha_{v_{(\ell)}} \equiv \alpha - c^{-1}(av_{(\ell)})^\cdot = 0$. Using Eqs. (169)–(171) we have

$$\ddot{\delta}_{(i)v_{(\ell)}} + 2H\dot{\delta}_{(i)v_{(\ell)}} - 4\pi G\sum_j \varrho_{(j)}\delta_{(j)v_{(j)}} = \frac{12\pi G}{c}aH\sum_j \varrho_{(j)}(v_{(\ell)} - v_{(j)}). \tag{176}$$

The terms in right-hand sides of Eqs. (175) and (176) look like relativistic correction terms present even to the linear order based on the variable $\delta_{(i)v_{(\ell)}}$. Since no such correction terms appear in Eq. (173) based on the variable $\delta_{(i)v}$, the relativistic correction terms in Eq. (176) can be regarded as being caused by a complicated hypersurface (gauge) choice.

From this comparison we notice that even for the individual component $v = 0$ gauge condition (we will simply call it the comoving gauge) produces Newtonian equations for $\delta_{(i)}$ to the linear order.

Exact solutions

Assuming $K = 0$, we can identify Newtonian perturbation variables as

$$\delta \equiv \delta_v, \qquad \kappa_v \equiv -\frac{1}{a}\nabla\cdot\mathbf{u}, \qquad \mathbf{u} \equiv -c\nabla v_\chi,$$
$$\delta\Phi \equiv -c^2\varphi_\chi, \qquad \delta_i \equiv \delta_{(i)v}, \qquad \mathbf{u}_i \equiv -c\nabla v_{(i)\chi}. \tag{177}$$

Equations (156)–(158), (160), and (171) become

$$\dot{\delta} = -\frac{1}{a}\nabla\cdot\mathbf{u}, \tag{178}$$

$$\dot{\mathbf{u}} + H\mathbf{u} = -\frac{1}{a}\nabla\delta\Phi, \tag{179}$$

$$\dot{\delta}_i = -\frac{1}{a}\nabla\cdot\mathbf{u}_i, \tag{180}$$

$$\dot{\mathbf{u}}_i + H\mathbf{u}_i = -\frac{1}{a}\nabla\delta\Phi, \tag{181}$$

$$\frac{\Delta}{a^2}\delta\Phi = 4\pi G\varrho\delta = 4\pi G\sum_j \varrho_j\delta_j. \tag{182}$$

Under the identification in Eq. (177) these equations are valid in both Newton's and Einstein's gravity theories. Equations (178), (179), and (182), and Eqs. (180)–(182), respectively, give

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\varrho\delta = \frac{1}{a^2 H}\left[a^2 H^2\left(\frac{\delta}{H}\right)^{\cdot}\right]^{\cdot} = 0, \quad (183)$$

$$\ddot{\delta}_i + 2H\dot{\delta}_i - 4\pi G\sum_j \varrho_j\delta_j = 0. \quad (184)$$

Equation (183) has an exact solution

$$\delta(\mathbf{x}, t) = H\left[c_g(\mathbf{x})\int^t \frac{dt}{a^2 H^2} + c_d(\mathbf{x})\right], \quad (185)$$

where c_g and c_d are integration constants indicating the relatively growing and decaying solutions, respectively, in an expanding phase; we do not consider the lower bound of integration which is absorbed to the c_d mode. Equations (183)–(185), and the solution in Eq. (185) are valid considering general K and Λ in the background world model. Equation (182) can be solved to give

$$\delta\Phi = -G\varrho a^2\int\frac{\delta(\mathbf{x}', t)}{|\mathbf{x}' - \mathbf{x}|}d^3x'. \quad (186)$$

From Eqs. (178), (179), and (182) we can show [13]

$$\mathbf{u} = -a\left(\frac{\nabla\delta\Phi}{4\pi G\varrho a^2}\right)^{\cdot} + \frac{1}{a}\mathbf{D}(\mathbf{x}), \qquad \nabla\cdot\mathbf{D} \equiv 0, \quad (187)$$

where the $\mathbf{D}$ term is the solution of the homogeneous part of Eq. (179); it decouples from the density inhomogeneity and corresponds to the peculiar velocity in the background world model. Since the $\mathbf{D}$ term is not connected to the density inhomogeneity and simply decays, we may ignore it to the linear order.

Now, for the individual component, from Eqs. (180) and (181) we have

$$\mathbf{u}_i = -a\left(\frac{\nabla\delta\Phi}{4\pi G\varrho a^2}\right)^{\cdot} + \frac{1}{a}\nabla d_i(\mathbf{x}) + \frac{1}{a}\mathbf{D}_i(\mathbf{x}), \qquad \nabla\cdot\mathbf{D}_i \equiv 0, \quad (188)$$

$$\delta_i = \delta + c_i(\mathbf{x}) - \Delta d_i(\mathbf{x})\int^t\frac{dt'}{a^2(t')}, \quad (189)$$

with

$$\sum_j \varrho_j d_j \equiv 0 \equiv \sum_j \varrho_j c_j. \quad (190)$$

The c_i and d_i are the two isocurvature-type ($\delta = 0$, thus $\delta\Phi = 0$) solutions. It happens that the relatively decaying isocurvature-type solution, i.e., d_i mode, temporally behaves the same as the peculiar velocity in the background, i.e., $\mathbf{D}_i$ modes. The relatively growing isocurvature-type solution c_i does not contribute to the $\mathbf{u}_i$, see Eq. (180). From Eqs. (187) and (188) we have

$$\mathbf{u}_i - \mathbf{u} = \frac{1}{a}[\nabla d_i(\mathbf{x}) + \mathbf{D}_i(\mathbf{x}) - \mathbf{D}(\mathbf{x})], \quad (191)$$

which simply decays; the $\mathbf{D}_i$ and $\mathbf{D}$ solutions are divergence free and decoupled from the density perturbation, and are the peculiar velocity perturbations present in the background world model.

C. Comoving gauge

To the linear order, only in the hypersurface condition $v = 0$ (the comoving temporal gauge) the density perturbation equations can be presented in the Newtonian form. Thus, we take

$$v \equiv 0, \quad (192)$$

even to the second order. We take $\gamma \equiv 0$ in Eq. (149) as the spatial gauge condition. Equation (154) gives

$$\alpha = -\frac{1}{2a^2}\chi^{,\alpha}\chi_{,\alpha} - \sum_j\frac{\mu_{(j)}}{\mu}\left[\frac{1}{2}v^{\alpha}_{(j)}v_{(j)\alpha} + \Delta^{-1}\nabla_\alpha(v^{\alpha}_{(j)}v^{\beta}_{(j)|\beta})\right]. \quad (193)$$

Using this, Eqs. (150)–(153) give

$$\dot{\delta} - \kappa = -\frac{c}{a^2}\delta_{,\alpha}\chi^{,\alpha} + \delta\kappa + \frac{1}{2}H\sum_j\mu_{(j)}v^{\alpha}_{(j)}v_{(j)\alpha} + 3H\sum_j\frac{\mu_{(j)}}{\mu}\Delta^{-1}\nabla_\alpha(v^{\alpha}_{(j)}v^{\beta}_{(j)|\beta}), \quad (194)$$

$$\dot{\kappa} + 2H\kappa - 4\pi G\varrho\delta = -\frac{c}{a^2}\kappa_{,\alpha}\chi^{,\alpha} + \frac{1}{3}\kappa^2 + \left(\dot{C}^{(t)\alpha\beta} + \frac{c}{a^2}\chi^{,\alpha|\beta}\right)\left(\dot{C}^{(t)}_{\alpha\beta} + \frac{c}{a^2}\chi_{,\alpha|\beta}\right) - \frac{1}{3}\left(c\frac{\Delta}{a^2}\chi\right)^2 + \frac{1}{2}\left(3\dot{H} + 8\pi G\varrho + c^2\frac{\Delta}{a^2}\right)\sum_j\mu_{(j)}v^{\alpha}_{(j)}v_{(j)\alpha} + \left(3\dot{H} + c^2\frac{\Delta}{a^2}\right)\sum_j\frac{\mu_{(j)}}{\mu}\Delta^{-1}\nabla_\alpha(v^{\alpha}_{(j)}v^{\beta}_{(j)|\beta}), \quad (195)$$

$$\dot{\delta}_{(i)} - \kappa + c\frac{1}{a}[(1 + \delta_{(i)})v^{\alpha}_{(i)}]_{|\alpha} = -\frac{c}{a^2}\delta_{(i),\alpha}\chi^{,\alpha} + \delta_{(i)}\kappa + Hv_{(i)\alpha}v^{\alpha}_{(i)} - \frac{c}{a}(\varphi^{,\alpha}v_{(i)\alpha} - 2\varphi v^{\alpha}_{(i)|\alpha} - 2v^{\alpha|\beta}_{(i)}C^{(t)}_{\alpha\beta}) + \frac{3}{2}H\sum_j\mu_{(j)}v^{\alpha}_{(j)}v_{(j)\alpha} + 3H\sum_j\frac{\mu_{(j)}}{\mu}\Delta^{-1}\nabla_\alpha(v^{\alpha}_{(j)}v^{\beta}_{(j)|\beta}), \quad (196)$$

$$\frac{1}{a}[a(1+\delta_{(i)})v_{(i)\alpha}]^{\cdot} = -\frac{c}{a^2}(v_{(i)\beta}\chi^{,\beta})_{,\alpha} + \kappa v_{(i)\alpha} - \frac{c}{a}(v_{(i)\alpha}v^{\beta}_{(i)})_{|\beta} + \frac{c}{2a}\sum_j \mu_{(j)}\nabla_\alpha(v^{\beta}_{(j)}v_{(j)\beta}) + \frac{c}{a}\sum_j \frac{\mu_{(j)}}{\mu}\nabla_\alpha\Delta^{-1}\nabla_\gamma(v^{\gamma}_{(j)}v^{\beta}_{(j)|\beta}). \tag{197}$$

From Eqs. (194) and (195), and Eqs. (195)–(197) we can derive, respectively,

$$\frac{1}{a^2}\left[a^2\left(\dot{\delta} + \frac{c}{a^2}\delta_{,\alpha}\chi^{,\alpha}\right)\right]^{\cdot} - 4\pi G\varrho\delta(1+\delta) = -\frac{c}{a^2}\kappa_{,\alpha}\chi^{,\alpha} + \frac{4}{3}\kappa^2 + \left(\dot{C}^{(t)\alpha\beta} + \frac{c}{a^2}\chi^{,\alpha|\beta}\right)\times\left(\dot{C}^{(t)}_{\alpha\beta} + \frac{c}{a^2}\chi_{,\alpha|\beta}\right) - \frac{1}{3}\left(c\frac{\Delta}{a^2}\chi\right)^2 + \left(2\dot{H} + 4\pi G\varrho + \frac{1}{2}c^2\frac{\Delta}{a^2}\right)\sum_j \mu_{(j)}v^{\alpha}_{(j)}v_{(j)\alpha} + \left(6\dot{H} + c^2\frac{\Delta}{a^2}\right)\sum_j \frac{\mu_{(j)}}{\mu}\Delta^{-1}\nabla_\alpha(v^{\alpha}_{(j)}v^{\beta}_{(j)|\beta}), \tag{198}$$

$$\frac{1}{a^2}\left[a^2\left(\dot{\delta}_{(i)} + \frac{c}{a^2}\delta_{(i),\alpha}\chi^{,\alpha}\right)\right]^{\cdot} - 4\pi G\varrho\delta(1+\delta_{(i)}) = -\frac{c}{a^2}\kappa_{,\alpha}\chi^{,\alpha} + \frac{4}{3}\kappa^2 + \left(\dot{C}^{(t)\alpha\beta} + \frac{c}{a^2}\chi^{,\alpha|\beta}\right)\left(\dot{C}^{(t)}_{\alpha\beta} + \frac{c}{a^2}\chi_{,\alpha|\beta}\right) - \frac{1}{3}\left(c\frac{\Delta}{a^2}\chi\right)^2 - \frac{c}{a}[(\kappa+\dot{\varphi})^{,\alpha}v_{(i)\alpha} + 2(\kappa-\dot{\varphi})v^{\alpha}_{(i)|\alpha}] + c^2\frac{\Delta}{a^3}(v_{(i)\alpha}\chi^{,\alpha}) + \frac{c^2}{a^2}(v^{\alpha}_{(i)}v^{\beta}_{(i)})_{|\alpha\beta} + \dot{H}v_{(i)\alpha}v^{\alpha}_{(i)} + \frac{2c}{a}v^{\alpha|\beta}_{(i)}\dot{C}^{(t)}_{\alpha\beta} + (3\dot{H} + 4\pi G\varrho)\sum_j \mu_{(j)}v^{\alpha}_{(j)}v_{(j)\alpha} + 6\dot{H}\sum_j \frac{\mu_{(j)}}{\mu}\Delta^{-1}\nabla_\alpha(v^{\alpha}_{(j)}v^{\beta}_{(j)|\beta}). \tag{199}$$

Notice the $\mathcal{O}(v^{\alpha}_{(j)}v_{(j)\alpha})$ correction terms are present in Eqs. (194), (195), and (198) even in the collective component situation. Except for these $\mathcal{O}(v^{\alpha}_{(j)}v_{(j)\alpha})$ terms, the remaining parts of these equations coincide with the ones in the single-component situation.

D. Newtonian correspondence

For $K = 0$, to the linear order, we have

$$\chi_v \equiv \chi - av \equiv -av_\chi, \qquad \kappa_v = -c\frac{\Delta}{a^2}\chi_v = c\frac{\Delta}{a}v_\chi, \qquad v_{(i)v} \equiv v_{(i)} - v = v_{(i)\chi} - v_\chi, \qquad \dot{\varphi}_v = 0. \tag{200}$$

To the linear order we identify

$$\mathbf{u} \equiv -c\nabla v_\chi, \qquad \mathbf{u}_i \equiv -c\nabla v_{(i)\chi}. \tag{201}$$

Now, to the second order, we *identify* the Newtonian perturbation variables δ, δ_i, $\mathbf{u}$, and $\mathbf{u}_i$ as

$$\kappa_v \equiv -\frac{1}{a}\nabla\cdot\mathbf{u}, \qquad \chi_v \equiv \frac{a}{c}u, \qquad \mathbf{u} \equiv \nabla u, \qquad \mathbf{v}_{(i)v} \equiv \frac{1}{c}(\mathbf{u}_i - \mathbf{u}), \qquad \delta \equiv \delta_v, \qquad \delta_i \equiv \delta_{(i)v}. \tag{202}$$

Using these identifications Eqs. (194)–(199) can be written as

$$\dot{\delta} + \frac{1}{a}\nabla\cdot\mathbf{u} = -\frac{1}{a}\nabla\cdot(\delta\mathbf{u}) + H\sum_j \frac{\varrho_j}{\varrho}\frac{1}{c^2}\left\{\frac{1}{2}|\mathbf{u}_j - \mathbf{u}|^2 + 3\Delta^{-1}\nabla\cdot[(\mathbf{u}_j - \mathbf{u})\nabla\cdot(\mathbf{u}_j - \mathbf{u})]\right\}, \tag{203}$$

$$\frac{1}{a}\nabla\cdot(\dot{\mathbf{u}} + H\mathbf{u}) + 4\pi G\varrho\delta = -\frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) - \dot{C}^{(t)\alpha\beta}\left(\frac{2}{a}u_{\alpha|\beta} + \dot{C}^{(t)}_{\alpha\beta}\right) + \sum_j \frac{\varrho_j}{\varrho}\frac{1}{c^2}\left\{\frac{1}{2}\left(4\pi G\varrho - c^2\frac{\Delta}{a^2}\right)|\mathbf{u}_j - \mathbf{u}|^2 + \left(12\pi G\varrho - c^2\frac{\Delta}{a^2}\right)\Delta^{-1}\nabla\cdot[(\mathbf{u}_j - \mathbf{u})\nabla\cdot(\mathbf{u}_j - \mathbf{u})]\right\}, \tag{204}$$

$$\dot{\delta}_i + \frac{1}{a}\nabla\cdot\mathbf{u}_i = -\frac{1}{a}\nabla\cdot(\delta_i\mathbf{u}_i) + \frac{1}{a}[2\varphi\nabla\cdot(\mathbf{u}_i-\mathbf{u}) - (\mathbf{u}_i-\mathbf{u})\cdot\nabla\varphi + 2(u_i^\alpha - u^\alpha)^{|\beta}C^{(t)}_{\alpha\beta}] + H\frac{1}{c^2}|\mathbf{u}_i-\mathbf{u}|^2 + 3H\sum_j\frac{\varrho_j}{\varrho}\frac{1}{c^2}\left\{\frac{1}{2}|\mathbf{u}_j-\mathbf{u}|^2 + \Delta^{-1}\nabla\cdot[(\mathbf{u}_j-\mathbf{u})\nabla\cdot(\mathbf{u}_j-\mathbf{u})]\right\}, \tag{205}$$

$$\frac{1}{a}\nabla\cdot(\dot{\mathbf{u}}_i + H\mathbf{u}_i) + 4\pi G\varrho\delta = -\frac{1}{a^2}\nabla\cdot(\mathbf{u}_i\cdot\nabla\mathbf{u}_i) - \dot{C}^{(t)\alpha\beta}\left(\frac{2}{a}u_{\alpha|\beta} + \dot{C}^{(t)}_{\alpha\beta}\right) + 4\pi G\sum_j\varrho_j\frac{1}{c^2}\left\{\frac{1}{2}|\mathbf{u}_j-\mathbf{u}|^2 + 3\Delta^{-1}\nabla\cdot[(\mathbf{u}_j-\mathbf{u})\nabla\cdot(\mathbf{u}_j-\mathbf{u})]\right\}, \tag{206}$$

$$\frac{1}{a^2}(a^2\dot{\delta})^{\cdot} - 4\pi G\varrho\delta = -\frac{1}{a^2}[a\nabla\cdot(\delta\mathbf{u})]^{\cdot} + \frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) + \dot{C}^{(t)\alpha\beta}\left(\frac{2}{a}u_{\alpha|\beta} + \dot{C}^{(t)}_{\alpha\beta}\right) - \sum_j\frac{\varrho_j}{\varrho}\frac{1}{c^2}\left\{\left(4\pi G\varrho - \frac{c^2}{2}\frac{\Delta}{a^2}\right)|\mathbf{u}_j - \mathbf{u}|^2 + \left(24\pi G\varrho - c^2\frac{\Delta}{a^2}\right)\Delta^{-1}\nabla\cdot[(\mathbf{u}_j-\mathbf{u})\nabla\cdot(\mathbf{u}_j-\mathbf{u})]\right\}, \tag{207}$$

$$\frac{1}{a^2}(a^2\dot{\delta}_i)^{\cdot} - 4\pi G\varrho\delta = -\frac{1}{a^2}[a\nabla\cdot(\delta_i\mathbf{u}_i)]^{\cdot} + \frac{1}{a^2}\nabla\cdot(\mathbf{u}_i\cdot\nabla\mathbf{u}_i) + \dot{C}^{(t)}_{\alpha\beta}\left(\frac{2}{a}u_i^{\alpha|\beta} + \dot{C}^{(t)\alpha\beta}\right) + \frac{1}{a^2}\{\Delta[\mathbf{u}\cdot(\mathbf{u}_i-\mathbf{u})] - \nabla\cdot[(\mathbf{u}_i-\mathbf{u})\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla(\mathbf{u}_i-\mathbf{u})]\} - \frac{4\pi G\varrho}{c^2}|\mathbf{u}_i-\mathbf{u}|^2 - 8\pi G\sum_j\varrho_j\frac{1}{c^2}\{|\mathbf{u}_j-\mathbf{u}|^2 + 3\Delta^{-1}\nabla\cdot[(\mathbf{u}_j-\mathbf{u})\nabla\cdot(\mathbf{u}_j-\mathbf{u})]\}. \tag{208}$$

In Sec. VIIB1 we have shown that, to the linear order, $(\mathbf{u}_i - \mathbf{u})$ simply decays ($\propto a^{-1}$) in an expanding phase. From Eq. (191) we have

$$\mathbf{u}_i - \mathbf{u} = \frac{1}{a}[\nabla d_i(\mathbf{x}) + \mathbf{D}_i(\mathbf{x}) - \mathbf{D}(\mathbf{x})], \quad \sum_j\varrho_j d_j \equiv 0, \qquad \nabla\cdot\mathbf{D} \equiv 0 \equiv \nabla\cdot\mathbf{D}_i. \tag{209}$$

Thus, $\mathbf{u}_i - \mathbf{u}$ simply decays in an expanding background. If we *ignore* these contributions from the velocity differences, except for the presence of the tensor-type perturbation, Eq. (208) coincides exactly with the zero-pressure limit of the Newtonian result in Eq. (16). Terms involving $\mathbf{u}_j - \mathbf{u}$ terms in Eq. (208) are relativistic correction terms which vanish for a single-component case leading to Eq. (207); similar terms in Eq. (207) also vanishes in the single-component case.

Ignoring quadratic combination of $(\mathbf{u}_i - \mathbf{u})$ terms, we have

$$\dot{\delta} + \frac{1}{a}\nabla\cdot\mathbf{u} = -\frac{1}{a}\nabla\cdot(\delta\mathbf{u}), \tag{210}$$

$$\frac{1}{a}\nabla\cdot(\dot{\mathbf{u}} + H\mathbf{u}) + 4\pi G\varrho\delta = -\frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) - \dot{C}^{(t)\alpha\beta}\left(\frac{2}{a}u_{\alpha|\beta} + \dot{C}^{(t)}_{\alpha\beta}\right), \tag{211}$$

$$\frac{1}{a^2}(a^2\dot{\delta})^{\cdot} - 4\pi G\varrho\delta = -\frac{1}{a^2}[a\nabla\cdot(\delta\mathbf{u})]^{\cdot} + \frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) + \dot{C}^{(t)\alpha\beta}\left(\frac{2}{a}u_{\alpha|\beta} + \dot{C}^{(t)}_{\alpha\beta}\right), \tag{212}$$

and

$$\dot{\delta}_i + \frac{1}{a}\nabla\cdot\mathbf{u}_i = -\frac{1}{a}\nabla\cdot(\delta_i\mathbf{u}_i) + \frac{1}{a}[2\varphi\nabla\cdot(\mathbf{u}_i-\mathbf{u}) - (\mathbf{u}_i-\mathbf{u})\cdot\nabla\varphi + 2(u_i^\alpha - u^\alpha)^{|\beta}C^{(t)}_{\alpha\beta}], \tag{213}$$

$$\frac{1}{a}\nabla\cdot(\dot{\mathbf{u}}_i + H\mathbf{u}_i) + 4\pi G\varrho\delta = -\frac{1}{a^2}\nabla\cdot(\mathbf{u}_i\cdot\nabla\mathbf{u}_i) - \dot{C}^{(t)\alpha\beta}\left(\frac{2}{a}u_{\alpha|\beta} + \dot{C}^{(t)}_{\alpha\beta}\right), \tag{214}$$

$$\frac{1}{a^2}(a^2\dot{\delta}_i)^{\cdot} - 4\pi G\varrho\delta = -\frac{1}{a^2}[a\nabla\cdot(\delta_i\mathbf{u}_i)]^{\cdot} + \frac{1}{a^2}\nabla\cdot(\mathbf{u}_i\cdot\nabla\mathbf{u}_i) + \dot{C}^{(t)}_{\alpha\beta}\left(\frac{2}{a}u_i^{\alpha|\beta} + \dot{C}^{(t)\alpha\beta}\right) + \frac{1}{a^2}\{\Delta[\mathbf{u}\cdot(\mathbf{u}_i-\mathbf{u})] - \nabla\cdot[(\mathbf{u}_i-\mathbf{u})\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla(\mathbf{u}_i-\mathbf{u})]\}. \tag{215}$$

Equations (210)–(212) coincide with the density and velocity perturbation equations of a single-component medium [19]; thus, except for the contribution from gravitational waves, these equations coincide with ones in the Newtonian context.

If we further ignore $(\mathbf{u}_i - \mathbf{u})$ terms appearing in the pure second-order combinations, Eqs. (213)–(215) become

$$\dot{\delta}_i + \frac{1}{a}\nabla\cdot\mathbf{u}_i = -\frac{1}{a}\nabla\cdot(\delta_i\mathbf{u}_i), \tag{216}$$

$$\frac{1}{a}\nabla\cdot(\dot{\mathbf{u}}_i + H\mathbf{u}_i) + 4\pi G\varrho\delta = -\frac{1}{a^2}\nabla\cdot(\mathbf{u}_i\cdot\nabla\mathbf{u}_i) - \dot{C}^{(t)\alpha\beta}\left(\frac{2}{a}u_{\alpha|\beta} + \dot{C}^{(t)}_{\alpha\beta}\right), \tag{217}$$

$$\frac{1}{a^2}(a^2\dot{\delta}_i)^{\cdot} - 4\pi G\varrho\delta = -\frac{1}{a^2}[a\nabla\cdot(\delta_i\mathbf{u}_i)]^{\cdot} + \frac{1}{a^2}\nabla\cdot(\mathbf{u}_i\cdot\nabla\mathbf{u}_i) + \dot{C}^{(t)}_{\alpha\beta}\left(\frac{2}{a}u_i^{\alpha|\beta} + \dot{C}^{(t)\alpha\beta}\right). \tag{218}$$

Notice that, by ignoring i indices, Eqs. (216)–(218) coincide with Eqs. (210)–(212). In this context, except for the contribution from gravitational waves, the above equations coincide exactly with ones in the Newtonian context even in the multicomponent case; compare with Eqs. (8), (9), and (16) without pressure. In the single-component situation such a relativistic/Newtonian correspondence to the second order was shown in [18,19]. In the present case, the same equation valid in the single component is now valid in the multicomponent case for the collective fluid variables. This justifies our identifications of Newtonian perturbation variables in Eq. (202).

For convenient reference, in Table II we summarize symbols used in this section; for the remaining variable, like the metric perturbation variables, see Table I.

VIII. EFFECTS OF CURVATURE

We *consider* a single zero-pressure, irrotational fluid. We *take* the temporal comoving gauge and the spatial C gauge, thus

$$v \equiv 0 \equiv \gamma. \tag{219}$$

The remaining variables in these gauge conditions are fully gauge invariant. In the presence of background curvature the basic equations are presented in Eqs. (122)–(124) for nonvanishing pressure, or Eqs. (196)–(199) for zero-pressure multiple component fluids. By setting pressures equal to zero in Eqs. (122)–(124), or from Eqs. (195)–(198), we have

TABLE II. Symbols used in Sec. VII.

Symbol	Definition	Equation
μ	Energy density of collective component ($\equiv \varrho c^2$ plus internal energy density)	(62)
$\delta\mu$	Perturbed energy density of collective component	(63)
$\mu_{(i)}$	Energy density of ith component	(59)
$\delta\mu_{(i)}$	Perturbed energy density of ith component	(59)
δ	Relative density perturbations of collective component ($\equiv \delta\mu/\mu$)	
	Newtonian relative density perturbations of collective component ($\equiv \delta\varrho/\varrho$)	
$\delta_{(i)}$	Relative density perturbations of ith component ($\equiv \delta\mu_{(i)}/\mu_{(i)}$)	
δ_i	Newtonian relative density perturbations of ith component ($\equiv \delta\varrho_i/\varrho_i$)	(177)
v_α	Spatial component of the collective four-vector $\tilde{u}_a$	
v	Scalar part of v_α; Temporal comoving gauge sets $v \equiv 0$	(84)
$v_{(i)\alpha}$	Spatial component of the ith four-vector $\tilde{u}_{(i)a}$	(86)
φ_v	A gauge-invariant combination using φ and v which becomes φ in the comoving gauge ($v \equiv 0$)	
φ_χ	A gauge-invariant combination using φ and χ which becomes φ in the zero-shear gauge ($\chi \equiv 0$)	
v_χ	A gauge-invariant combination using v and χ which becomes v in the zero-shear gauge	
$v_{(i)\chi}$	A gauge-invariant combination using $v_{(i)}$ and χ which becomes $v_{(i)}$ in the zero-shear gauge	
χ_v	A gauge-invariant combination using χ and v which becomes χ in the comoving gauge	
α_χ	A gauge-invariant combination using α and χ which becomes α in the zero-shear gauge	
δ_v	A gauge-invariant combination using δ and v which becomes δ in the comoving gauge	
$\delta_{(i)v}$	A gauge-invariant combination using $\delta_{(i)}$ and v which becomes $\delta_{(i)}$ in the comoving gauge	
$\delta_{(i)v_{(\ell)}}$	A gauge-invariant combination using $\delta_{(i)}$ and $v_{(\ell)}$ which becomes $\delta_{(i)}$ in comoving gauge of ℓth component ($v_{(\ell)} \equiv 0$)	
$\delta\Phi$	Perturbed Newtonian gravitational potential	(177)
$\mathbf{u}$	Relativistic velocity perturbation of collective component identified from κ_v	(177)
	Coincide with Newtonian velocity perturbation of collective component to the second order	
$\mathbf{u}_i$	Relativistic velocity perturbation of ith component identified from $\mathbf{v}_{(i)v}$ and $\mathbf{u}$	(177)
	Coincide with Newtonian velocity perturbation of ith component to the second order	

$$\dot{\delta} - \kappa = -\frac{c}{a^2}\delta_{,\alpha}\chi^{\alpha} + \delta\kappa, \tag{220}$$

$$\dot{\kappa} + 2H\kappa - 4\pi G\varrho\delta = -\frac{c}{a^2}\kappa_{,\alpha}\chi^{\alpha} + \frac{1}{3}\kappa^2 - \frac{1}{3}\left(c\frac{\Delta}{a^2}\chi\right)^2 + \left(\dot{C}^{(t)\alpha\beta} + \frac{c}{a^2}\chi^{\alpha|\beta}\right) \times \left(\dot{C}^{(t)}_{\alpha\beta} + \frac{c}{a^2}\chi_{,\alpha|\beta}\right), \tag{221}$$

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\varrho\delta = 4\pi G\varrho\delta^2 - \frac{c}{a^2}(\delta_{,\alpha}\chi^{\alpha})^{\cdot} - \frac{c}{a^2}\kappa_{,\alpha}\chi^{\alpha} + \frac{4}{3}\kappa^2 - \frac{1}{3}\left(c\frac{\Delta}{a^2}\chi\right)^2 + \left(\dot{C}^{(t)\alpha\beta} + \frac{c}{a^2}\chi^{\alpha|\beta}\right) \times \left(\dot{C}^{(t)}_{\alpha\beta} + \frac{c}{a^2}\chi_{,\alpha|\beta}\right), \tag{222}$$

$$c\frac{\Delta + 3K}{a^2}\chi + \kappa = 0, \tag{223}$$

where Eq. (223) is valid to the linear order. Compared with the situation with vanishing curvature, the effects of curvature in the above perturbed set of equations appear only in the linear-order relation between κ and χ in Eq. (223).

A. Newtonian correspondence

Considering the successful Newtonian correspondence to the linear order even in the presence of the background curvature, we *assume* the identification in Eq. (127) is valid to the second order. Then, to the linear order, from Eq. (223) we have

$$\kappa \equiv -\frac{1}{a}\nabla\cdot\mathbf{u} \equiv -\frac{\Delta}{a}u = -c\frac{\Delta + 3K}{a^2}\chi, \tag{224}$$

where $\mathbf{u} \equiv \nabla u$ and $\chi = \chi_v = -av_{\chi}$. Thus, to the linear order, formally we have

$$\frac{c}{a}\chi = \left(1 - \frac{3K}{\Delta + 3K}\right)u. \tag{225}$$

In the presence of curvature, the scalar-type perturbation can be handled by solving Eqs. (220), (221), and (223) together with the identifications made above. We can formally separate the effects of the pure curvature contribution as follows. Using Eqs. (223)–(225), Eqs. (220) and (221) become

$$\dot{\delta} + \frac{1}{a}\nabla\cdot\mathbf{u} = -\frac{1}{a}\nabla\cdot(\delta\mathbf{u}) + \frac{1}{a}(\nabla\delta)\cdot\nabla\left(\frac{3K}{\Delta + 3K}u\right), \tag{226}$$

$$\frac{1}{a}\nabla\cdot(\dot{\mathbf{u}} + H\mathbf{u}) + 4\pi G\varrho\delta = -\frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) - \dot{C}^{(t)\alpha\beta}\left(\dot{C}^{(t)}_{\alpha\beta} + \frac{2}{a}u_{,\alpha|\beta}\right) + \frac{1}{a^2}\nabla\cdot\left[\left(\frac{3K}{\Delta + 3K}\mathbf{u}\right)\cdot\nabla\mathbf{u}\right] - \frac{1}{3}\frac{1}{a^2} \times \left(\frac{3K}{\Delta + 3K}\nabla\cdot\mathbf{u}\right)\left(\frac{2\Delta + 3K}{\Delta + 3K}\nabla\cdot\mathbf{u}\right) + \frac{1}{a}\left(\frac{3K}{\Delta + 3K}u\right)^{,\alpha|\beta}\left[2\dot{C}^{(t)}_{\alpha\beta} + \frac{1}{a}\left(\frac{\Delta}{\Delta + 3K}u\right)_{,\alpha|\beta}\right]. \tag{227}$$

Notice that Eqs. (226) and (227) do not have the K term explicitly to the linear order. Notice also that to the linear order these equations coincide with the Newtonian ones in Eqs. (8)–(10) without using the gravitational potential. Therefore, to the linear order in a single-component zero-pressure fluid, we have relativistic/Newtonian correspondence of the equations for density and velocity perturbations without using the gravitational potential or the metric perturbations in the presence of background spatial curvature. The same is true for the vector-type perturbation: to the linear order we have the relativistic/Newtonian correspondence of the vector-type velocity perturbation equation even in the presence of background spatial curvature and the anisotropic stress, see below Eq. (237).

From Eq. (79), the K term can be written as

$$K = \left(\frac{aH}{c}\right)^2(\Omega_t - 1), \qquad \Omega_t \equiv \Omega + \Omega_{\Lambda}, \qquad \Omega \equiv \frac{8\pi G\varrho}{3H^2}, \qquad \Omega_{\Lambda} \equiv \frac{\Lambda c^2}{3H^2}. \tag{228}$$

Symbols used in this section are the same as the ones summarized in Table I.

IX. EFFECTS OF VECTOR-TYPE PERTURBATION

The spatial C-gauge sets

$$\gamma \equiv 0 \equiv C^{(v)}_{\alpha}. \tag{229}$$

The remaining variables under this gauge condition are completely free of the spatial gauge modes and have unique spatially gauge-invariant counterparts. If we simultaneously take *any* temporal gauge which also removes the temporal gauge mode completely, all the remaining variables have corresponding unique gauge-invariant counterparts. The above statements are true to all orders in perturbations, see Sec. VI of [18]. From Eq. (82) we have

$$\beta \equiv \frac{1}{a}\chi, \qquad B^{(v)}_{\alpha} \equiv \Psi^{(v)}_{\alpha}. \tag{230}$$

Thus, we have

$$B_\alpha = \frac{1}{a}\chi_{,\alpha} + \Psi^{(v)}_\alpha, \qquad C_{\alpha\beta} \equiv \varphi g^{(3)}_{\alpha\beta} + C^{(t)}_{\alpha\beta}. \quad (231)$$

As the temporal comoving gauge we set

$$v \equiv 0. \quad (232)$$

As we mentioned, the remaining variables under these gauge conditions are completely free of the gauge modes and have unique gauge-invariant counterparts to all orders in perturbation, see [18].

A. Linear perturbations

To the linear order, the three types of perturbations decouple, and evolve independently. The rotational perturbation is described by Eqs. (206)–(209) in [18]:

$$\frac{\Delta + 2K}{2a^2}\Psi^{(v)}_\alpha + \frac{8\pi G}{c^4}(\mu + p)v^{(v)}_\alpha = 0, \quad (233)$$

$$\frac{[a^4(\mu + p)v^{(v)}_\alpha]^\cdot}{a^4(\mu + p)} = -c\frac{\Delta + 2K}{2a^2}\frac{\Pi^{(v)}_\alpha}{\mu + p}, \quad (234)$$

$$\dot{\Psi}^{(v)}_\alpha + 2H\Psi^{(v)}_\alpha = \frac{8\pi G}{c^3}\Pi^{(v)}_\alpha, \quad (235)$$

where the last equation follows from the first two.

Later in Eq. (258) we will introduce the Newtonian vector-type velocity perturbation as $u^{(v)}_\alpha \equiv cv^{(v)}_\alpha$. Using this, in the zero-pressure limit, but keeping the anisotropic stress, Eq. (234) becomes

$$\dot{u}^{(v)}_\alpha + Hu^{(v)}_\alpha = -\frac{\Delta + 2K}{2a^2\varrho}\Pi^{(v)}_\alpha. \quad (236)$$

If we ignore the anisotropic stress, this equation is the same as the Newtonian one in Eq. (15). In the presence of the stress Eq. (9) for a single-component fluid becomes

$$\dot{u}_\alpha + Hu_\alpha + \frac{1}{a}u^\beta\nabla_\beta u_\alpha = -\frac{1}{a\varrho}(\nabla_\alpha\delta p + \nabla_\beta\Pi^\beta_\alpha) + \frac{1}{a}\nabla_\alpha\delta\Phi. \quad (237)$$

This equation is derived in the Newtonian limit (more precisely, in the cosmological zeroth post-Newtonian order) of Einstein's gravity in [16], and coincides with the Navier-Stokes equation known in the Newtonian context; see Eq. (72) or (107) in [16]. Using Eq. (85) and $u_\alpha = u_{,\alpha} + u^{(v)}_\alpha$ in Eq. (258), to the linear order, Eq. (237) leads to Eq. (236) exactly including the K term. Therefore, to the linear order the rotational velocity perturbation has exact Newtonian correspondence even in the presence of background spatial curvature. Concerning Eq. (233), however, in the Newtonian context we do not use the vectorial velocity potential for the rotational velocity field. Introducing the vectorial velocity potential $\Psi^{(v)}_\alpha$ as in Eq. (233) even in the Newtonian context does not contradict Newtonian gravity, *ad hoc*, though. As we have shown, a similar situation occurred in the scalar-type perturbation of a single-component fluid. Without using the metric perturbations, the equations for the density and velocity perturbations in Einstein's gravity coincide with the Newtonian ones without using the gravitational potential; this is true even in the presence of the background spatial curvature, see below Eq. (227).

Compared with Bardeen's notation in [12], we have

$$\begin{aligned} B^{(v)}_\alpha &= B^{(1)}Q^{(1)}_\alpha, \qquad C^{(v)}_\alpha = -\frac{1}{k}H^{(1)}_T Q^{(1)}_\alpha, \\ \Psi^{(v)}_\alpha &= \Psi Q^{(1)}_\alpha \equiv \left(B^{(1)} - \frac{a}{k}\dot{H}^{(1)}_T\right)Q^{(1)}_\alpha, \\ v^{(v)}_\alpha &= v_c Q^{(1)}_\alpha \equiv (v^{(1)} - B^{(1)})Q^{(1)}_\alpha, \\ v^{(v)}_\alpha + \Psi^{(v)}_\alpha &= v^{(1)}_s Q^{(1)}_\alpha \equiv \left(v^{(1)} - \frac{a}{k}\dot{H}^{(1)}_T\right)Q^{(1)}_\alpha, \end{aligned} \quad (238)$$

where $Q^{(1)}_\alpha$ is a vector-type harmonic function. Bardeen's $v^{(1)}_c$ and $v^{(1)}_s$, thus our $v^{(v)}_\alpha$ and $v^{(v)}_\alpha + \Psi^{(v)}_\alpha$, are related to the vorticity and the shear, respectively. From Eq. (65), to the linear order, we have

$$\begin{aligned} \tilde{\omega}_{\alpha\beta} &= av^{(v)}_{[\alpha|\beta]}, \\ \tilde{\sigma}_{\alpha\beta} &= a\left[-\left(\nabla_\alpha\nabla_\beta - \frac{1}{3}g^{(3)}_{\alpha\beta}\Delta\right)\left(v - \frac{1}{a}\chi\right) + \left(v^{(v)}_{(\alpha} + \Psi^{(v)}_{(\alpha}\right)_{|\beta)} + C^{(t)\prime}_{\alpha\beta}\right]. \end{aligned} \quad (239)$$

Bardeen called $\Psi^{(v)}_\alpha$ a "frame-dragging potential." The difference between $v^{(1)}_c$ and $v^{(1)}_s$ is crucially important to show Mach's principle including the linear-order rotational perturbation, see Schmid's work in [29]. We introduce

$$\sigma^{(v)}_\alpha \equiv v^{(v)}_\alpha + \Psi^{(v)}_\alpha = v^{(1)}_s Q^{(1)}_\alpha, \qquad v^{(v)}_\alpha = v^{(1)}_c Q^{(1)}_\alpha. \quad (240)$$

From Eq. (54) we have

$$\tilde{T}^0_\alpha = (\mu + p)v^{(v)}_\alpha, \qquad \tilde{T}^\alpha_0 = -(\mu + p)\sigma^{(v)}_\alpha. \quad (241)$$

Using $\sigma^{(v)}_\alpha$, Eq. (233) can be written as

$$\left(\frac{\Delta - 2K}{a^2} + \frac{4}{c^2}\dot{H}\right)\Psi^{(v)}_\alpha = -\frac{16\pi G}{c^4}(\mu + p)\sigma^{(v)}_\alpha, \quad (242)$$

where we used the background equations in Eqs. (78) and (79). For $K = 0$, this equation reduces to Eq. (40) in [29]; the presence of the $\dot{H}$ term in the left-hand side leads to an exponential suppression of the dragging of the axes of gyroscopes by matter beyond the horizon.

In the absence of anisotropic stress which can act as a sink or source of the angular momentum, we have

$$\text{angular momentum} \propto a^4(\mu + p)v^{(v)}_\alpha \propto a^2\Psi^{(v)}_\alpha \propto \text{constant in time}. \quad (243)$$

Thus, for vanishing anisotropic stress, we have

$$v_\alpha^{(v)} \propto \frac{1}{a^4(\mu + p)}, \qquad \Psi_\alpha^{(v)} \propto \frac{1}{a^2}. \tag{244}$$

In the zero-pressure limit we have

$$v_\alpha^{(v)} \propto \frac{1}{a}, \qquad \Psi_\alpha^{(v)} \propto \frac{1}{a^2}. \tag{245}$$

B. Second-order perturbations with pressure

We consider a single-component situation with general pressure and stresses. We set $K \equiv 0$. To the linear order we use

$$\chi \equiv \chi_v \equiv -av_\chi, \qquad \kappa \equiv \kappa_v = c\frac{\Delta}{a}v_\chi. \tag{246}$$

The scalar-type perturbation is described by Eqs. (70) and (72) which give

$$\begin{aligned}\dot\delta + 3H(c_s^2 - w)\delta + 3H\frac{e}{\mu} - (1+w)\kappa = {}& -3H(1+w)\Big[\alpha - \frac{1}{2}\alpha^2 + \frac{1}{2}(v_\chi{}^{,\alpha} - \Psi^{(v)\alpha})(v_{\chi,\alpha} - \Psi_\alpha^{(v)})\Big] \\ & + \frac{c}{a}\delta_{,\alpha}(v_\chi{}^{,\alpha} - \Psi^{(v)\alpha}) + \Big(\delta + \frac{\delta p}{\mu}\Big)\Big(c\frac{\Delta}{a}v_\chi - 3H\alpha\Big) + (1+w)\alpha c\frac{\Delta}{a}v_\chi \\ & + (1+w)v_\alpha^{(v)}\Big[(1 - 3c_s^2)Hv^{(v)\alpha} + \frac{c}{\mu+p}\frac{\Delta}{a^2}\Pi^{(v)\alpha}\Big] \\ & + \frac{c}{a}(1+w)[-(2\alpha+\varphi)_{,\alpha}v^{(v)\alpha} + 2C^{(t)\alpha\beta}v^{(v)}_{\beta|\alpha}] - \dot\varphi\frac{1}{\mu}\Pi^\alpha_\alpha - \Big(\dot C^{(t)}_{\alpha\beta} - \frac{c}{a}v_{\chi,\alpha|\beta} \\ & + \frac{c}{a}\Psi^{(v)}_{\alpha|\beta}\Big)\frac{1}{\mu}\Pi^{\alpha\beta} - \frac{1}{a\mu}[(\delta\mu + \delta p)_{,\alpha}v^{(v)\alpha} + (\Pi^{\alpha\beta}v^{(v)}_\beta)_{|\alpha}],\end{aligned} \tag{247}$$

$$\begin{aligned}\dot\kappa + 2H\kappa - \frac{4\pi G}{c^2}(\delta\mu + 3\delta p) = {}& -\Big(3\dot H + c^2\frac{\Delta}{a^2}\Big)\Big[\alpha + \frac{1}{2}(v_\chi{}^{,\alpha} - \Psi^{(v)\alpha})(v_{\chi,\alpha} - \Psi_\alpha^{(v)})\Big] - 2H\alpha c\frac{\Delta}{a}v_\chi \\ & + \frac{4\pi G}{c^2}(\delta\mu + 3\delta p)\alpha + \frac{3}{2}\dot H\alpha^2 + (\alpha + 2\varphi)c^2\frac{\Delta}{a^2}\alpha + \frac{c^2}{a^2}\alpha^{,\alpha}(\alpha - \varphi)_{,\alpha} + \frac{2c^2}{a^2}C^{(t)\alpha\beta}\alpha_{,\alpha|\beta} \\ & + \frac{8\pi G}{c^2}(\mu + p)v^{(v)\alpha}v^{(v)}_\alpha + (v_\chi{}^{,\alpha} - \Psi^{(v)\alpha})\Big(c^2\frac{\Delta}{a^2}v_\chi\Big)_{,\alpha} + \Big(\dot C^{(t)\alpha\beta} - \frac{c}{a}v_\chi{}^{,\alpha|\beta} + \frac{c}{a}\Psi^{(v)(\alpha|\beta)}\Big) \\ & \times\Big(\dot C^{(t)}_{\alpha\beta} - \frac{c}{a}v_{\chi,\alpha|\beta} + \frac{c}{a}\Psi^{(v)}_{\alpha|\beta}\Big).\end{aligned} \tag{248}$$

The vector-type perturbation is described by Eq. (73) which gives

$$\begin{aligned}\frac{1}{a^4}[a^4(\mu+p)v_\alpha^{(v)}]^{\cdot} + \frac{c}{a}(\mu+p)\alpha_{,\alpha} + \frac{c}{a}(\delta p_{,\alpha} + \Pi^\beta_{\alpha|\beta}) = {}& \frac{c}{a}\Big\{-\alpha\delta p + \frac{1}{2}(\mu+p)[\alpha^2 - (v_\chi{}^{,\beta} - \Psi^{(v)\beta})(v_{\chi,\beta} - \Psi^{(v)}_\beta)]\Big\}_{,\alpha} \\ & - \frac{c}{a}\alpha_{,\alpha}\delta\mu + (\mu+p)\Big(c\frac{\Delta}{a}v_\chi - 3H\alpha\Big)v^{(v)}_\alpha \\ & - \frac{c}{a}(\mu+p)[v^{(v)}_\beta(-v_\chi{}^{,\beta} + \Psi^{(v)\beta})_{|\alpha} + v^{(v)}_{\alpha|\beta}(v^{(v)\beta} - v_\chi{}^{,\beta} \\ & + \Psi^{(v)\beta})] + \frac{c}{a}[2\varphi\Pi^\beta_{\alpha|\beta} - \varphi_{,\beta}\Pi^\beta_\alpha + \varphi_{,\alpha}\Pi^\beta_\beta - (\alpha\Pi^\beta_\alpha)_{|\beta} \\ & + 2C^{(t)\beta\gamma}\Pi_{\alpha\beta|\gamma} + C^{(t)\gamma}{}_{\beta|\alpha}\Pi^\beta_\gamma] \\ & - \frac{1}{a^4}\{a^4[(\delta\mu + \delta p)v^{(v)}_\alpha + \Pi^\beta_\alpha v^{(v)}_\beta]\}^{\cdot} \equiv \mu c A_\alpha.\end{aligned} \tag{249}$$

We have $[A_\alpha] = L^{-1}$. From this we have

$$(\mu + p)\alpha + \delta p + \frac{2}{3}\frac{\Delta}{a^2}\Pi = a\mu\Delta^{-1}\nabla\cdot\mathbf{A}, \tag{250}$$

$$\frac{1}{a^4}[a^4(\mu+p)v^{(v)}_\alpha]^{\cdot} + c\frac{\Delta}{2a^2}\Pi^{(v)}_\alpha = \mu c[A_\alpha - (\Delta^{-1}\nabla\cdot\mathbf{A})_{,\alpha}]. \tag{251}$$

Equation for the tensor-type perturbation (gravitational waves) follows from Eqs. (71) and (109).

C. Zero-pressure case

In the zero-pressure limit, Eqs. (249) and (250) give

$$\alpha = a\Delta^{-1}\nabla\cdot\mathbf{A}, \tag{252}$$

$$aA_\alpha \equiv -\tfrac{1}{2}[(v_\chi{}^{,\beta} - \Psi^{(v)\beta})(v_{\chi,\beta} - \Psi^{(v)}_\beta)]_{,\alpha} + v^{(v)}_{\alpha|\beta}(v_\chi{}^{,\beta} - \Psi^{(v)\beta}) + v^{(v)}_\beta(v_\chi{}^{,\beta} - \Psi^{(v)\beta})_{|\alpha} - v^{(v)}_{\alpha|\beta}v^{(v)\beta}, \tag{253}$$

thus, α is purely second order, and

$$\alpha + \tfrac{1}{2}(v_\chi^{,\beta} - \Psi^{(v)\beta})(v_{\chi,\beta} - \Psi^{(v)}_\beta) = \Delta^{-1}\nabla^\alpha[v^{(v)}_{\alpha|\beta}(v_\chi^{,\beta} - \Psi^{(v)\beta}) + v^{(v)}_\beta(v_\chi^{,\beta} - \Psi^{(v)\beta})_{|\alpha} - v^{(v)}_{\alpha|\beta}v^{(v)\beta}]. \tag{254}$$

Equations (247), (248), and (251) give

$$\dot\delta - \kappa = \frac{c}{a}[\delta(v_\chi{}^{,\alpha} - \Psi^{(v)\alpha} - v^{(v)\alpha})]_{|\alpha} + Hv^{(v)}_\alpha v^{(v)\alpha} + \frac{c}{a}(-\varphi^{,\alpha}v^{(v)}_\alpha + 2C^{(t)\alpha\beta}v^{(v)}_{\beta|\alpha}) - 3H\Delta^{-1}\nabla^\alpha[v^{(v)}_{\alpha|\beta}(v_\chi{}^{,\beta} - \Psi^{(v)\beta}) + v^{(v)}_\beta(v_\chi{}^{,\beta} - \Psi^{(v)\beta})_{|\alpha} - v^{(v)}_{\alpha|\beta}v^{(v)\beta}], \tag{255}$$

$$\dot\kappa + 2H\kappa - 4\pi G\varrho\delta = \frac{c^2}{a^2}[(v_\chi{}^{,\alpha} - \Psi^{(v)\alpha})(v_\chi{}^{,\beta} - \Psi^{(v)\beta})_{|\alpha}]_{|\beta} + \dot C^{(t)\alpha\beta}\bigg[\dot C^{(t)}_{\alpha\beta} - \frac{2c}{a}(v_{\chi,\alpha|\beta} - \Psi^{(v)}_{\alpha|\beta})\bigg] + 8\pi G\varrho v^{(v)\alpha}v^{(v)}_\alpha - \bigg(3\dot H + c^2\frac{\Delta}{a^2}\bigg) \times \Delta^{-1}\nabla^\alpha[v^{(v)}_{\alpha|\beta}(v_\chi{}^{,\beta} - \Psi^{(v)\beta}) + v^{(v)}_\beta(v_\chi{}^{,\beta} - \Psi^{(v)\beta})_{|\alpha} - v^{(v)}_{\alpha|\beta}v^{(v)\beta}], \tag{256}$$

$$\dot v^{(v)}_\alpha + Hv^{(v)}_\alpha = \frac{c}{a}[v^{(v)}_{\alpha|\beta}(v_\chi{}^{,\beta} - \Psi^{(v)\beta}) + v^{(v)}_\beta(v_\chi{}^{,\beta} - \Psi^{(v)\beta})_{|\alpha} - v^{(v)}_{\alpha|\beta}v^{(v)\beta}] - \frac{c}{a}\nabla_\alpha\Delta^{-1}\nabla^\gamma[v^{(v)}_{\gamma|\beta}(v_\chi{}^{,\beta} - \Psi^{(v)\beta}) + v^{(v)}_\beta(v_\chi{}^{,\beta} - \Psi^{(v)\beta})_{|\gamma} - v^{(v)}_{\gamma|\beta}v^{(v)\beta}]. \tag{257}$$

D. Newtonian correspondence

In order to compare with Newtonian equations we continue identifying κ as in Eq. (127) to the second order. To the linear order we identify

$$\mathbf{u} \equiv c\mathbf{v} \equiv c(-\nabla v_\chi + \mathbf{v}^{(v)}) \equiv \nabla u + \mathbf{u}^{(v)}, \tag{258}$$

thus

$$u \equiv -cv_\chi, \qquad \mathbf{u}^{(v)} \equiv c\mathbf{v}^{(v)}. \tag{259}$$

We introduce the following notations

$$U_\alpha \equiv u_\alpha + c\Psi^{(v)}_\alpha, \qquad \tilde U_\alpha \equiv u_{,\alpha} + c\Psi^{(v)}_\alpha. \tag{260}$$

As mentioned below Eq. (238), to the linear order, u_α and U_α are related to the vorticity and the shear, respectively. Equations (255)–(257) become

$$\dot\delta + \frac{1}{a}\nabla\cdot\mathbf{u} = -\frac{1}{a}\nabla\cdot(\delta\mathbf{U}) + \frac{2}{a}C^{(t)\alpha\beta}u^{(v)}_{\alpha|\beta} - \frac{1}{a}\mathbf{u}^{(v)}\cdot\nabla\varphi + \frac{1}{c^2}H[\mathbf{u}^{(v)}\cdot\mathbf{u}^{(v)} + 3\Delta^{-1}\nabla^\alpha(u^{(v)}_{\alpha|\beta}U^\beta + u^{(v)\beta}\tilde U_{\beta|\alpha})], \tag{261}$$

$$\frac{1}{a}\nabla\cdot(\dot{\mathbf{u}} + H\mathbf{u}) + 4\pi G\varrho\delta = -\frac{1}{a^2}\nabla\cdot(\mathbf{U}\cdot\nabla\mathbf{U}) - \dot C^{(t)\alpha\beta}\bigg(\dot C^{(t)}_{\alpha\beta} + \frac{2}{a}\tilde U_{\alpha|\beta}\bigg) + \frac{8\pi G\varrho}{c^2}\bigg[-\mathbf{u}^{(v)}\cdot\mathbf{u}^{(v)} + \frac{3}{2}\Delta^{-1}\nabla^\alpha(u^{(v)}_{\alpha|\beta}U^\beta + u^{(v)\beta}\tilde U_{\beta|\alpha})\bigg], \tag{262}$$

$$\dot{\mathbf{u}}^{(v)} + H\mathbf{u}^{(v)} = -\frac{1}{a}[\mathbf{U}\cdot\nabla\mathbf{u} - \nabla\Delta^{-1}\nabla\cdot(\mathbf{U}\cdot\nabla\mathbf{u})] - \frac{1}{a}\times[U^\beta c\Psi^{(v)}_{\beta|\alpha} - \nabla_\alpha\Delta^{-1}\nabla^\gamma(U^\beta c\Psi^{(v)}_{\beta|\gamma})]. \tag{263}$$

From Eq. (263) we notice that the tensor-type perturbation does not affect the vector-type perturbation to the second order. The pure scalar-type perturbation also cannot generate the vector-type perturbation to the second order; the same is true in the Newtonian case, see below Eq. (15). The above equations show that the scalar- and vector-type perturbations are nontrivially connected to the second order compared with the Newtonian situation.

From Eq. (233), to the linear order, we have

$$v^{(v)}_\alpha = \frac{1}{6}\bigg(\frac{ck}{aH}\bigg)^2\Psi^{(v)}_\alpha, \tag{264}$$

where k is a comoving wave number with $\Delta \equiv -k^2$, thus $[k] = L^{-1}$. Since $(ck)/(aH)\sim$ (visual horizon)/(scale), we have

$$\begin{aligned} &\text{far inside horizon: } v^{(v)}_\alpha \gg \Psi^{(v)}_\alpha, \\ &\mathbf{U} \simeq \mathbf{u} = \nabla u + \mathbf{u}^{(v)}, \qquad \tilde{\mathbf{U}} \simeq \nabla u, \\ &\text{far outside horizon: } v^{(v)}_\alpha \ll \Psi^{(v)}_\alpha, \\ &\mathbf{U} \simeq \tilde{\mathbf{U}} = u_{,\alpha} + c\Psi^{(v)}_\alpha. \end{aligned} \tag{265}$$

Apparently, contributions of vector-type perturbation to the second order depend on the visual-horizon scale.

Far inside the horizon, we can ignore $c\Psi^{(v)}_\alpha$ compared with $u^{(v)}_\alpha$. In the matter dominated era we have

$$\varphi_v = \frac{5}{3}\varphi_\chi = \frac{5}{3}\frac{a^2}{k^2}\frac{4\pi G\varrho}{c^2}\delta_v = \frac{5}{2}\left(\frac{aH}{kc}\right)^2\delta_v, \quad (266)$$

where we used Eqs. (293), (329), and (330) of [18]. Thus, the third term in the right-hand side of Eq. (261) is $(aH/kc)^2$-order smaller than the first term. The fourth (and last) term in the right-hand side of Eq. (261) is $(aH/kc)[\mathbf{u}^{(v)}/(c\delta)]$-order smaller than the first term. The third (and last) term in the right-hand side of Eq. (262) is also $(aH/kc)^2$-order smaller than the first term. Thus, Eqs. (261)–(263) give

$$\dot\delta + \frac{1}{a}\nabla\cdot\mathbf{u} = -\frac{1}{a}\nabla\cdot(\delta\mathbf{u}) + \frac{2}{a}C^{(t)\alpha\beta}u^{(v)}_{\alpha|\beta}, \quad (267)$$

$$\frac{1}{a}\nabla\cdot(\dot{\mathbf{u}} + H\mathbf{u}) + 4\pi G\varrho\delta = -\frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) - \dot C^{(t)\alpha\beta}\left[\dot C^{(t)}_{\alpha\beta} + \frac{2}{a}(u_{,\alpha|\beta} + c\Psi^{(v)}_{\alpha|\beta})\right], \quad (268)$$

$$\dot{\mathbf{u}}^{(v)} + H\mathbf{u}^{(v)} = -\frac{1}{a}[\mathbf{u}\cdot\nabla\mathbf{u} - \nabla\Delta^{-1}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u})]. \quad (269)$$

Thus, if we could ignore the tensor-type combinations in Eqs. (267) and (268), Eqs. (267)–(269) coincide exactly with the Newtonian equations: see Eqs. (8), (12), and (15) ignoring the pressure terms and the subindices i. The vector-tensor combinations in Eq. (267) and (268) are new relativistic contributions of the vector-type perturbations; compare these two equations with Eqs. (17) and (18) which are valid in the absence of the vector-type perturbations. Notice the form of the last term $u_{,\alpha|\beta} + c\Psi^{(v)}_{\alpha|\beta}$ in Eq. (268) which subtly differs from the expression $u_{\alpha|\beta}$ in Eq. (18).

Contributions from the vector-type perturbation become more complicated near and outside the horizon scale. Examination of Eqs. (261)–(263) shows that the presence of the vector-type metric perturbation $\Psi^{(v)}_\alpha$, the scalar-type curvature perturbation φ, and the tensor-type perturbation $C^{(t)}_{\alpha\beta}$ coupled with the vector-type perturbation give additional effects. Considering the pure decaying nature of vector-type perturbations to the linear order, unless we have exotic sources of the rotational perturbations in the later epoch, it is likely that the vector-type contribution (generated in the early universe) to the second order is negligible near the horizon scale.

We summarize symbols used in this section in Table III; for the remaining symbols, see Table I.

E. Pure vector-type perturbations

In Sec. VII-E of [18] we have considered a situation with pure vector-type perturbation. As the analysis was made based on the fluid quantities in the normal frame, in the following we present the case based on the fluid quantities in the energy frame. Considering only the vector-type perturbation of a fluid, Eq. (73) gives

$$\frac{1}{a^4(\mu+p)}\left\{a^4\left[(\mu+p)v^{(v)}_\alpha + \frac{1}{a}\Pi^{(v)}_{(\alpha|\beta)}v^{(v)\beta}\right]\right\}^{\cdot} = -\frac{c(\Delta+2K)\Pi^{(v)}_\alpha}{2a^2(\mu+p)} - \frac{c}{a}\left[(v^{(v)\beta} + \Psi^{(v)|\beta})\times(v^{(v)}_{\alpha|\beta} + \Psi^{(v)}_{\beta|\alpha}) - \frac{2}{3}v^{(v)\beta}v^{(v)}_{\beta|\alpha}\right]. \quad (270)$$

Thus, for $\Pi^{(v)}_\alpha = 0$ we have

$$\frac{1}{a^4(\mu+p)}[a^4(\mu+p)v^{(v)}_\alpha]^{\cdot} = -\frac{c}{a}\left[(v^{(v)\beta} + \Psi^{(v)|\beta})\times(v^{(v)}_{\alpha|\beta} + \Psi^{(v)}_{\beta|\alpha}) - \frac{2}{3}v^{(v)\beta}v^{(v)}_{\beta|\alpha}\right]. \quad (271)$$

This differs from Eq. (365) of [18] which is due to the difference in the frame choice. The momentum constraint equation in Eq. (101) of [18] becomes

TABLE III. Symbols used in Sec. IX.

Symbol	Definition	Equation
$B^{(v)}_\alpha$	Transverse vector-type metric perturbation	(50) and (81)
$C^{(v)}_\alpha$	Transverse vector-type metric perturbation	(50) and (81)
$\Psi^{(v)}_\alpha$	Gauge-invariant combination of $B^{(v)}_\alpha$ and $C^{(v)}_\alpha$	(82)
$\Pi^{(v)}_\alpha$	Transverse vector part of the anisotropic stress $\tilde\pi_{\alpha\beta}$	(52) and (85)
u_α	Newtonian velocity ($\equiv cv_\alpha$)	(84) and (258)
$u^{(v)}_\alpha$	Transverse vector part of u_α	(258) and (259)
U_α	u_α plus $c\Psi^{(v)}_\alpha$	(260)
$\tilde U_\alpha$	$u_{,\alpha}$ plus $c\Psi^{(v)}_\alpha$	(260)

$$\frac{\Delta + 2K}{2a^2}\Psi_\alpha^{(v)} + \frac{8\pi G}{c^4}(\mu + p)v_\alpha^{(v)} = -\frac{8\pi G}{c^4}\frac{1}{a}\Pi^{(v)}_{(\alpha|\beta)}v^{(v)\beta}. \quad (272)$$

Under the gauge transformation, from Eq. (66) we have

$$\hat{v}_\alpha^{(v)} = v_\alpha^{(v)} - v_\beta^{(v)}\xi^{(v)\beta}{}_{,\alpha} - v^{(v)}_{\alpha,\beta}\xi^{(v)\beta}. \quad (273)$$

To the linear order, from Eq. (230) of [18] we have $\hat{B}_\alpha^{(v)} = B_\alpha^{(v)} + \xi_\alpha^{(v)\prime}$. We consider a gauge transformation from the C gauge ($C_\alpha^{(v)} \equiv 0$, without hat) to the B gauge ($B_\alpha^{(v)} \equiv 0$, with hat). We have $\hat{B}_\alpha^{(v)} \equiv 0$, and $B_\alpha^{(v)}|_{C\text{-gauge}} = -\xi_\alpha^{(v)\prime}$. Thus,

$$\xi_\alpha^{(v)} = -\int^\eta B_\alpha^{(v)}|_{C\text{-gauge}}d\eta = -a^2\Psi_\alpha^{(v)}\int^\eta \frac{d\eta}{a^2}, \quad (274)$$

where we used $B_\alpha^{(v)}|_{C\text{-gauge}} = \Psi_\alpha^{(v)} \propto a^{-2}$. Thus, Eq. (273) gives

$$v_\alpha^{(v)}|_{B\text{-gauge}} = \Big[v_\alpha^{(v)} + (v_\beta^{(v)}\Psi^{(v)\beta}{}_{,\alpha} + v^{(v)}_{\alpha,\beta}\Psi^{(v)\beta})a^2\int^\eta\frac{d\eta}{a^2}\Big]_{C\text{-gauge}}. \quad (275)$$

X. EQUATIONS WITH FIELDS

A. A minimally coupled scalar field

Equations in the case of a minimally coupled scalar field are presented in Eqs. (112)–(114) of [18]. The equation of motion in Eq. (112) and the full Einstein's equations in Eqs. (99)–(105) expressed using the normal-frame fluid quantities together with the normal-frame fluid quantities for the scalar field in Eq. (114) all in [18] provide a complete set of equations we need to the second order. The fluid quantities in Eq. (114) of [18] are presented in the normal-frame four-vector and it is convenient to know the conventionally used fluid quantities which are based on the energy-frame four-vector. These latter quantities can be read from Eqs. (88) and (114) of [18] and Eq. (84) as

$$\begin{aligned} Q_\alpha^{N(\phi)} &= -\frac{1}{a}[\dot\phi\delta\phi_{,\alpha} + \delta\phi_{,\alpha}(\delta\dot\phi - \dot\phi A)] \\ &= (\mu^{(\phi)} + p^{(\phi)})(V_\alpha^{(\phi)} - B_\alpha + AB_\alpha + 2V^{(\phi)\beta}C_{\alpha\beta}) + (\delta\mu^{(\phi)} + \delta p^{(\phi)})(V_\alpha^{(\phi)} - B_\alpha) \\ &= (\mu^{(\phi)} + p^{(\phi)} + \delta\mu^{(\phi)} + \delta p^{(\phi)})(-v_{,\alpha}^{(\phi)} + v_\alpha^{(\phi,v)}), \\ \delta\mu^{(\phi)} &= \delta\mu^{N(\phi)} - \frac{1}{a^2}\delta\phi^{,\alpha}\delta\phi_{,\alpha}, \qquad \delta p^{(\phi)} = \delta p^{N(\phi)} - \frac{1}{3a^2}\delta\phi^{,\alpha}\delta\phi_{,\alpha}, \\ \Pi_{\alpha\beta}^{(\phi)} &= \Pi_{\alpha\beta}^{N(\phi)} - \frac{1}{a^2}\Big(\delta\phi_{,\alpha}\delta\phi_{,\beta} - \frac{1}{3}g^{(3)}_{\alpha\beta}\delta\phi^{,\gamma}\delta\phi_{,\gamma}\Big). \end{aligned} \quad (276)$$

Thus, using Eq. (114) of [18] we have

$$\begin{aligned} v^{(\phi)} &= \frac{1}{a\dot\phi}\delta\phi - \frac{1}{a\dot\phi^2}\Delta^{-1}\nabla^\alpha[(\delta\dot\phi - A\dot\phi)\delta\phi_{,\alpha}], \qquad v_\alpha^{(\phi,v)} = \frac{1}{a\dot\phi^2}\{(\delta\dot\phi - \dot\phi A)\delta\phi_{,\alpha} - \nabla_\alpha\Delta^{-1}\nabla^\beta[(\delta\dot\phi - \dot\phi A)\delta\phi_{,\beta}]\}, \\ \delta\mu^{(\phi)} &= \dot\phi\delta\dot\phi - \dot\phi^2 A + V_{,\phi}\delta\phi + \frac{1}{2}\delta\dot\phi^2 - \frac{1}{2a^2}\delta\phi^{,\alpha}\delta\phi_{,\alpha} + \frac{1}{2}V_{,\phi\phi}\delta\phi^2 - 2\dot\phi\delta\dot\phi A + \frac{1}{a}\dot\phi\delta\phi_{,\alpha}B^\alpha + 2\dot\phi^2A^2 \\ &\quad - \frac{1}{2}\dot\phi^2B^\alpha B_\alpha, \\ \delta p^{(\phi)} &= \dot\phi\delta\dot\phi - \dot\phi^2 A - V_{,\phi}\delta\phi + \frac{1}{2}\delta\dot\phi^2 - \frac{1}{2a^2}\delta\phi^{,\alpha}\delta\phi_{,\alpha} - \frac{1}{2}V_{,\phi\phi}\delta\phi^2 - 2\dot\phi\delta\dot\phi A + \frac{1}{a}\dot\phi\delta\phi_{,\alpha}B^\alpha + 2\dot\phi^2A^2 \\ &\quad - \frac{1}{2}\dot\phi^2B^\alpha B_\alpha, \\ \Pi_{\alpha\beta}^{(\phi)} &= 0. \end{aligned} \quad (277)$$

Notice that no anisotropic stress is caused by a minimally coupled scalar field even to the second order in perturbations. The uniform-field gauge takes $\delta\phi \equiv 0$ as a temporal gauge (slicing) condition to the second order in perturbation. The uniform-field gauge gives $v^{(\phi)} = 0$ which is the comoving gauge, and vice versa. Thus,

$$\delta\phi = 0 \leftrightarrow v^{(\phi)} = 0. \quad (278)$$

We also have

$$\delta\mu^{(\phi)} - \delta p^{(\phi)} = 2V_{,\phi}\delta\phi + V_{,\phi\phi}\delta\phi^2, \quad (279)$$

and under the uniform-field gauge we have

$$\delta\mu^{(\phi)}_{\delta\phi} = \delta p^{(\phi)}_{\delta\phi} = -\dot{\phi}^2[A(1-2A) + \tfrac{1}{2}B^\alpha B_\alpha]_{\delta\phi}. \quad (280)$$

Equation (277) apparently shows that the vector-type perturbation $v^{(\phi,v)}_\alpha$ does not vanish to the second order. However, the second-order quantities in the right-hand side depend on the temporal gauge condition for the scalar-type perturbations, and trivially vanish for the uniform-field gauge. We can also show that it vanishes for the uniform-density gauge or the uniform-pressure gauge where we have $\delta\dot{\phi} - \dot{\phi}A \propto \delta\phi$ to the linear order. In fact, we can show that the vector-type perturbation is not sourced by the scalar field to the second order by evaluating the rotational tensor $\tilde{\omega}_{\alpha\beta}$ in Eq. (65) for the scalar field: i.e., for a minimally coupled scalar field we have

$$\tilde{\omega}^{(\phi)}_{\alpha\beta} = 0, \quad (281)$$

to the second order. Thus, we conclude that a minimally coupled scalar field does *not* contribute to the rotational perturbation to the second order in perturbations. In fact, we can show that a single scalar field does not support vector-type perturbations to all orders in perturbations. As we take the energy frame, thus $\tilde{q}_a \equiv 0$, from Eq. (23) of [18] we can show

$$\tilde{u}_a = \frac{\tilde{\phi}_{,a}}{\sqrt{-\tilde{\phi}^{,c}\tilde{\phi}_{,c}}}. \quad (282)$$

Using the definition of the vorticity tensor in Eq. (33) we can show that $\tilde{\omega}_{ab} = 0$.

Using the fluid quantities in Eq. (277) we can handle the scalar field using our nonideal fluid formulation. The fluid equations in the energy frame, like Eqs. (67)–(73) remain valid in the case of the scalar field with the fluid quantities expressed as in Eq. (277). The anisotropic stress vanishes and the entropic perturbation e is given as

$$e^{(\phi)} \equiv \delta p^{(\phi)} - c^2_{(\phi)}\delta\mu^{(\phi)}, \qquad c^2_{(\phi)} \equiv \frac{\dot{p}^{(\phi)}}{\dot{\mu}^{(\phi)}} = \frac{\ddot{\phi} - V_{,\phi}}{\ddot{\phi} + V_{,\phi}}. \quad (283)$$

Under the comoving gauge $v \equiv 0$, using Eq. (280) we have

$$\delta p^{(\phi)}_v = \delta\mu^{(\phi)}_v, \qquad e^{(\phi)}_v = (1 - c^2_{(\phi)})\delta\mu^{(\phi)}_v, \quad (284)$$

which is a well-known relation in linear perturbation [27]; now we showed that the relation is valid to the second order in perturbations. A fluid formulation of the scalar field to the linear order is presented in [27,30]. Using Eq. (284) together with vanishing anisotropic stress, Eqs. (67)–(73) or Eqs. (95)–(101) provide the fluid formulation for a minimally coupled scalar field to the second order in perturbations. The perturbed equation of motion of the scalar field is presented in Eq. (112) of [18].

B. Minimally coupled scalar fields

In the case of multiple minimally coupled scalar fields, the equation of motions in Eq. (119) and the full Einstein's equations in Eqs. (99)–(105) expressed using the normal-frame fluid quantities together with the normal-frame fluid quantities for the scalar field in Eq. (121) all in [18] provide a complete set of equations we need to the second order. The fluid quantities in Eq. (121) of [18] are presented in the normal-frame four-vector and it is convenient to know the conventionally used fluid quantities which are based on the energy-frame four-vector. These latter quantities can be read from Eq. (121) of [18] and Eq. (56) as

$$\begin{aligned}
v^{(\phi)} &= \frac{1}{a\sum_n \dot{\phi}^2_{(n)}}\sum_k\Bigg\{\dot{\phi}_{(k)}\delta\phi_{(k)} + \Delta^{-1}\nabla^\alpha\Bigg[(\delta\dot{\phi}_{(k)} - \dot{\phi}_{(k)}A)\Bigg(\delta\phi_{(k)} - 2\dot{\phi}_{(k)}\frac{\sum_l\dot{\phi}_{(l)}\delta\phi_{(l)}}{\sum_m\dot{\phi}^2_{(m)}}\Bigg)_{,\alpha}\Bigg]\Bigg\},\\
v^{(\phi,v)}_\alpha &= -\frac{1}{a\sum_n\dot{\phi}^2_{(n)}}\sum_k\Bigg\{(\delta\dot{\phi}_{(k)} - \dot{\phi}_{(k)}A)\Bigg(\delta\phi_{(k)} - 2\dot{\phi}_{(k)}\frac{\sum_l\dot{\phi}_{(l)}\delta\phi_{(l)}}{\sum_m\dot{\phi}^2_{(m)}}\Bigg)_{,\alpha}\\
&\quad - \nabla_\alpha\Delta^{-1}\nabla^\beta\Bigg[(\delta\dot{\phi}_{(k)} - \dot{\phi}_{(k)}A)\Bigg(\delta\phi_{(k)} - 2\dot{\phi}_{(k)}\frac{\sum_l\dot{\phi}_{(l)}\delta\phi_{(l)}}{\sum_m\dot{\phi}^2_{(m)}}\Bigg)_{,\beta}\Bigg]\Bigg\},\\
\delta\mu^{(\phi)} &= \delta\mu^{N(\phi)} - \frac{1}{a^2\sum_m\dot{\phi}^2_{(m)}}\Bigg(\sum_k\dot{\phi}_{(k)}\delta\phi_{(k)}{}^{,\alpha}\Bigg)\Bigg(\sum_l\dot{\phi}_{(l)}\delta\phi_{(l),\alpha}\Bigg),\\
\delta p^{(\phi)} &= \delta p^{N(\phi)} - \frac{1}{3}\frac{1}{a^2\sum_m\dot{\phi}^2_{(m)}}\Bigg(\sum_k\dot{\phi}_{(k)}\delta\phi_{(k)}{}^{,\alpha}\Bigg)\Bigg(\sum_l\dot{\phi}_{(l)}\delta\phi_{(l),\alpha}\Bigg),\\
\Pi^{(\phi)}_{\alpha\beta} &= \Pi^{N(\phi)}_{\alpha\beta} - \frac{1}{a^2\sum_m\dot{\phi}^2_{(m)}}\Bigg[\Bigg(\sum_k\dot{\phi}_{(k)}\delta\phi_{(k),\alpha}\Bigg)\Bigg(\sum_l\dot{\phi}_{(l)}\delta\phi_{(l),\beta}\Bigg) - \frac{1}{3}g^{(3)}_{\alpha\beta}\Bigg(\sum_k\dot{\phi}_{(k)}\delta\phi_{(k)}{}^{,\gamma}\Bigg)\Bigg(\sum_l\dot{\phi}_{(l)}\delta\phi_{(l),\gamma}\Bigg)\Bigg],
\end{aligned} \quad (285)$$

where the normal-frame fluid quantities are presented in Eq. (121) of [18]. Thus, compared with the fluid quantities in the normal frame, the ones in the energy frame look rather more complicated. In a single field case we have $\Pi^{(\phi)}_{\alpha\beta} = 0$ in Eq. (277), whereas $\Pi^{(\phi)}_{\alpha\beta}$ in the normal frame does not vanish, see Eq. (114) of [18]. Apparently, in the multicomponent situation, $\Pi^{(\phi)}_{\alpha\beta}$ in the energy frame does not vanish and looks more complicated. Thus, in the multifield situation we had better use both the fluid quantities and Einstein's equations all expressed in the normal frame: these are Eqs. (99)–(105), and (121) in [18].

As the temporal gauge condition we can set any one field perturbation, say the specific ℓth one $\delta\phi_{(\ell)}$, equal to zero which might be called the uniform-$\phi_{(\ell)}$ gauge to the second order. This apparently differs from the comoving gauge which sets $v^{(\phi)} \equiv 0$. In the multicomponent situation it is nontrivial to take a gauge condition which makes $v^{(\phi,v)}_\alpha = 0$. The multiple scalar fields may source the vector-type perturbation to the second order in perturbations, see Eqs. (249) and (251). The scalar-, vector-, and tensor-type decomposition of the anisotropic stress $\Pi^{(\phi)}_{\alpha\beta}$ can be read by using decomposition formulae in Eq. (177) of [18].

In a multiple-field situation, it is *ad hoc* and cumbersome (if not impossible) to introduce an individual fluid quantity for each field variable even to the background and linear-order perturbations, see [31].

C. Generalized gravity case

In Sec. IV.D of [18] we presented the equation of motion and effective fluid quantities in a class of generalized gravity theories together with additional presence of fluids and fields to the second order. The equation of motion is in Eq. (128) of [18], and the full Einstein's equations in Eqs. (99)–(105) expressed using the normal-frame fluid quantities together with the normal-frame effective fluid quantities in Eq. (130) of [18] provide a complete set of equations we need to the second order. The effective fluid quantities in Eq. (130) of [18] are presented in the normal-frame four-vector and using Eq. (88) in [18] we can easily derive the effective fluid quantities based on the energy-frame four-vector. As in the multiple-field case in a previous subsection, by moving into the energy frame, the effective fluid quantities become more complicated compared with the ones in the normal frame. Thus, in these class of generalized gravity theories we better use both the fluid quantities and Einstein's equations all expressed in the normal frame: these are Eqs. (99)–(105), and (130) in [18].

XI. CURVATURE PERTURBATIONS AND LARGE-SCALE CONSERVATIONS

In the large-scale limit the spatial curvature perturbation φ in several different gauge conditions is known to remain constant in the expanding phase. Often the conservation properties are shown based on the first time derivative of the curvature perturbation. In order to show the conservation properties properly we have to construct the closed form second-order differential equations for the curvature perturbation. In the following we will derive such first-order and second-order differential equations for φ_v, φ_χ, φ_κ, and φ_δ. First we will derive equations for the linear perturbation including the background curvature and non-ideal fluid properties. Then we will derive equations for the second-order perturbation assuming a flat background; including the background curvature is trivial, though. We consider a single-component fluid.

A. Linear-order equations

We introduce a combination

$$\Phi \equiv \varphi_v - \frac{K/a^2}{4\pi G(\mu + p)}\varphi_\chi. \tag{286}$$

This combination was first introduced by Field and Shepley in [10]. From Eqs. (95), (97), and (99), Eqs. (95)–(97), (99), and (101), Eqs. (95) and (100), and Eqs. (95), (96), and (98), respectively, we can derive

$$\frac{H}{a}\left(\frac{a}{H}\varphi_\chi\right)^\cdot = \frac{4\pi G(\mu + p)}{H}\Phi - 8\pi G H\Pi, \tag{287}$$

$$\dot\Phi = \frac{Hc_s^2}{4\pi G(\mu + p)}\frac{\Delta}{a^2}\varphi_\chi - \frac{H}{\mu + p}\left(e + \frac{2}{3}\frac{\Delta}{a^2}\Pi\right), \tag{288}$$

$$\dot\varphi_\delta = \frac{\Delta}{3a}v_\chi - \frac{He}{\mu + p}, \tag{289}$$

$$\dot\varphi_\kappa = -\frac{H}{3\dot H + \Delta/a^2}\left[(1 + 3c_s^2)\frac{\Delta + 3K}{a^2}\varphi_\kappa - 12\pi G e\right] - \frac{\Delta}{3a^2}\chi_\kappa. \tag{290}$$

These equations were presented in [27,32].

We can derive closed form second-order differential equations for Φ, φ_χ, φ_δ, and φ_κ

$$\frac{\mu + p}{H}\left[\frac{H^2}{(\mu + p)a}\left(\frac{a}{H}\varphi_\chi\right)^\cdot + \frac{8\pi G H^2}{\mu + p}\Pi\right]^\cdot = c_s^2\frac{\Delta}{a^2}\varphi_\chi - 4\pi G\left(e + \frac{2}{3}\frac{\Delta}{a^2}\Pi\right), \tag{291}$$

$$\frac{H^2 c_s^2}{(\mu + p)a^3}\left[\frac{(\mu + p)a^3}{H^2 c_s^2}\dot\Phi + \frac{a^3}{Hc_s^2}\left(e + \frac{2}{3}\frac{\Delta}{a^2}\Pi\right)\right]^\cdot = c_s^2\frac{\Delta}{a^2}\left(\Phi - \frac{2H^2}{\mu + p}\Pi\right), \tag{292}$$

$$\frac{3\dot{H}+\Delta/a^2}{(\mu+p)a^3}\left[\frac{(\mu+p)a^3}{3\dot{H}+\Delta/a^2}\left(\dot{\varphi}_\delta+\frac{H}{\mu+p}e\right)\right]^{\cdot} = -\frac{4\pi G(\mu+p)+c_s^2(\Delta+3K)/a^2}{12\pi G(\mu+p)-(\Delta+3K)/a^2}\frac{\Delta}{a^2}\varphi_\delta + \frac{1}{\mu+p}\frac{\Delta}{a^2}\left[e+\frac{2}{3}\left(3\dot{H}+\frac{\Delta}{a^2}\right)\Pi\right], \quad (293)$$

$$\frac{1}{a^3}\left\{a^3\left[\dot{\varphi}_\kappa+\frac{H}{3\dot{H}+\Delta/a^2}\left((1+3c_s^2)\frac{\Delta+3K}{a^2}\varphi_\kappa-12\pi Ge\right)\right]\right\}^{\cdot} = -\frac{\Delta}{3a^2}\left[\varphi_\kappa-\frac{1}{3\dot{H}+\Delta/a^2}\times\left((1+3c_s^2)\frac{\Delta+3K}{a^2}\varphi_\kappa-12\pi Ge\right)+8\pi G\Pi\right]. \quad (294)$$

Equations (291) and (292) follow by combining Eqs. (287) and (288). Eqs. (293) and (294) follow from Eqs. (95), (96), (98), and (99), and Eqs. (95), (96), (98), and (99), respectively.

From Eqs. (96) and (97) we can derive

$$\varphi_\delta = \left[1-\frac{\Delta+3K}{12\pi G(\mu+p)a^2}\right]\varphi_\kappa = \varphi_v-\frac{\Delta+3K}{12\pi G(\mu+p)a^2}\varphi_\chi = \Phi-\frac{\Delta}{12\pi G(\mu+p)a^2}\varphi_\chi. \quad (295)$$

These relations were presented in [21]. In the large-scale limit and in near flat background, thus ignoring Δ and K terms, we have

$$\Phi \simeq \varphi_v \simeq \varphi_\delta \simeq \varphi_\kappa, \quad (296)$$

to the leading order in the large-scale expansion.

In the large-scale limit, ignoring Δ terms, in near flat background and for an ideal fluid case, thus setting $K=0$ and $e=0=\Pi$, Eqs. (291)–(294) give

$$\left(\frac{a}{H}\varphi_\chi\right)^{\cdot} \propto \frac{(\mu+p)a}{H^2}, \quad (297)$$

$$\dot{\varphi}_v \propto \frac{H^2c_s^2}{(\mu+p)a^3}, \quad (298)$$

$$\dot{\varphi}_\delta \propto \dot{\varphi}_\kappa \propto \frac{1}{a^3}. \quad (299)$$

Notice that if we simply ignore the Δ terms in the first-order differential equations in Eqs. (288)–(290) we simply have $\dot{\varphi}_v=\dot{\varphi}_\delta=\dot{\varphi}_\kappa=0$. In such a way we cannot recover the terms in the right-hand side of Eqs. (298) and (299); these terms lead to decaying solutions (in an expanding era) in the large-scale limit and are higher order in the large-scale expansion compared with the decaying solution of φ_χ, see below. From Eqs. (297)–(299) we have general large-scale asymptotic solutions

$$\varphi_\chi = 4\pi GC(\mathbf{x})\frac{H}{a}\int^t\frac{a(\mu+p)}{H^2}dt+d(\mathbf{x})\frac{H}{a}, \quad (300)$$

$$\varphi_v = C(\mathbf{x})+\frac{\Delta}{4\pi G}d(\mathbf{x})\int^t\frac{c_s^2H^2}{(\mu+p)a^3}dt, \quad (301)$$

$$\varphi_\delta=\varphi_\kappa = C(\mathbf{x})+\frac{\Delta}{3}d(\mathbf{x})\int^t\frac{dt}{a^3}, \quad (302)$$

where $C(\mathbf{x})$ and $d(\mathbf{x})$ are integration constants which correspond to the relatively growing and decaying modes, respectively, in an expanding phase; in a collapsing phase the roles are reversed. The coefficients are fixed using the relations in Eqs. (287), (288), and (295). Notice that for the C mode the relation in Eq. (296) is satisfied, and simply remains constant. For the d mode, φ_v, φ_δ, and φ_κ are $\Delta/(aH)^2$-order higher compared with the d mode of φ_χ. The φ_v is one of the well-known conserved quantity in the large scale even in the context of generalized gravity theories [27,33].

In order to evaluate the solutions to the second order in the next section, we need complete sets of linear-order solutions for different gauge conditions. For an ideal fluid, and for a minimally coupled scalar field such complete sets of solutions are presented in tabular forms in [30,32]. In the following we summarize such sets of solutions in an ideal fluid case for four different gauge conditions. From Table 8 of [32] we have

$$\begin{aligned}
&\varphi_\chi=-\alpha_\chi=C\left(1-\frac{H}{a}\int^t adt\right),\quad \delta_\chi=-\frac{2}{3}\frac{\kappa_\chi}{H}=-2C\frac{\dot{H}}{aH}\int^t adt,\quad v_\chi=-C\frac{1}{a^2}\int^t adt;\quad \varphi_v=C,\\
&H\chi_v=C\frac{H}{a}\int^t adt,\quad \delta_v=-\frac{1+w}{c_s^2}\alpha_v=-\frac{2}{3}\frac{\Delta}{a^2H^2}C\left(1-\frac{H}{a}\int^t adt\right),\quad \frac{\kappa_v}{H}=-\frac{\Delta}{a^2H^2}C\frac{H}{a}\int^t adt;\\
&\varphi_\delta=C,\quad H\chi_\delta=C\frac{H}{a}\int^t adt,\quad \frac{\kappa_\delta}{H}=3\alpha_\delta=-\frac{\Delta}{a^2H^2}C,\quad v_\delta=\frac{1}{3}\frac{\Delta}{a^2H^2}C\frac{H}{a\dot{H}}\left(1-\frac{H}{a}\int^t adt\right);\\
&\varphi_\kappa=C,\quad H\chi_\kappa=C\frac{H}{a}\int^t adt,\quad \delta_\kappa=-3\frac{1+w}{1+3c_s^2}\alpha_\kappa=-\frac{2}{3}\frac{\Delta}{a^2H^2}C,\quad v_\kappa=-\frac{1}{3}\frac{\Delta}{a^2\dot{H}}C\frac{1}{a^2}\int^t adt.
\end{aligned} \quad (303)$$

For corresponding sets of solutions for a minimally coupled scalar field, see Table 1 of [30]. Compared with the notation used in [32] we have $\gamma\Psi = -8\pi G(\mu + p)av$. The lower bounds of integration of solutions in Eq. (303) give behaviors of d modes. For solutions without integration, the d modes are $\Delta/(aH)^2$-order higher than the nonvanishing d mode, for example, see the solutions in Table 2 of [32]. Thus, the d modes are

$$\begin{aligned}
&\varphi_\chi = -\alpha_\chi = d\frac{H}{a}, \quad \delta_\chi = -\frac{2}{3}\frac{\kappa_\chi}{H} = 2d\frac{\dot H}{aH}, \quad v_\chi = d\frac{1}{a^2}; \quad \varphi_v = \frac{\Delta}{4\pi G}d(\mathbf{x})\int^t \frac{c_s^2H^2}{(\mu+p)a^3}dt,\\
&H\chi_v = -\frac{H}{a}d, \quad \delta_v = -\frac{1+w}{c_s^2}\alpha_v = -\frac{2}{3}\frac{\Delta}{a^2H^2}d\frac{H}{a}, \quad \frac{\kappa_v}{H} = \frac{\Delta}{a^3H}d; \quad \varphi_\delta = \frac{\Delta}{3}d(\mathbf{x})\int^t\frac{dt}{a^3},\\
&H\chi_\delta = -\frac{H}{a}d, \quad \frac{\kappa_\delta}{H} = -\frac{1}{3}\frac{\Delta^2}{a^2H^2}d\int^t\frac{dt}{a^3}, \quad \alpha_\delta = \frac{1}{9}\frac{\Delta^2}{a^2H^2}d\left(\frac{H}{a^3\dot H} - \int^t\frac{dt}{a^3}\right), \quad v_\delta = \frac{1}{3}\frac{\Delta}{a^4\dot H}d;\\
&\varphi_\kappa = \frac{\Delta}{3}d(\mathbf{x})\int^t\frac{dt}{a^3}, \quad H\chi_\kappa = -\frac{H}{a}d, \quad \delta_\kappa = -3\frac{1+w}{1+3c_s^2}\alpha_\kappa = -\frac{2}{9}\frac{\Delta^2}{a^2H^2}d\int^t\frac{dt}{a^3}, \quad v_\kappa = \frac{1}{3}\frac{\Delta}{a^4\dot H}d.
\end{aligned} \tag{304}$$

B. Second-order equations

We assume $K = 0$. From Eqs. (95), (97), and (99), Eqs. (95)–(97) and (101), Eqs. (95) and (100), and Eqs. (95), (96), and (98), respectively, we can derive

$$\frac{H}{a}\left[\frac{a}{H}(\varphi - H\chi)\right]^\cdot = \frac{4\pi G(\mu+p)}{H}(\varphi - aHv) - 8\pi GH\Pi + \frac{1}{3}(n_0 - n_2) - Hn_4, \tag{305}$$

$$(\varphi - aHv)^\cdot = \frac{Hc_s^2}{4\pi G(\mu+p)}\left[\frac{\Delta}{a^2}(\varphi - H\chi) - \frac{1}{4}n_1 + Hn_2\right] - \frac{H}{\mu+p}\left(e + \frac{2}{3}\frac{\Delta}{a^2}\Pi\right) + \frac{1}{3}(n_0 - n_2) - aHn_6, \tag{306}$$

$$\dot\varphi_\delta = \frac{\Delta}{3a}\left(v - \frac{1}{a}\chi\right) - \frac{He}{\mu+p} + \frac{1}{3}\left(n_0 + \frac{n_5}{\mu+p}\right), \tag{307}$$

$$\dot\varphi_\kappa = -\frac{H}{3\dot H + \Delta/a^2}\left[(1+3c_s^2)\frac{\Delta}{a^2}\varphi_\kappa - 12\pi Ge - \frac{1+3c_s^2}{4}n_1 - n_3\right] - \frac{\Delta}{3a^2}\chi + \frac{1}{3}n_0. \tag{308}$$

The perturbed-order variables in Eqs. (307) and (308) are evaluated in the uniform-density gauge ($\delta \equiv 0$), and the uniform-expansion gauge ($\kappa \equiv 0$), respectively. Equation (307) also follows from Eq. (41) evaluated to the second order.

We can derive closed form second-order differential equations for φ_v, φ_χ, φ_δ, and φ_κ

$$\frac{\mu+p}{H}\left\{\frac{H}{\mu+p}\left[\frac{H}{a}\left(\frac{a}{H}(\varphi - H\chi)\right)^\cdot + 8\pi GH\Pi - \frac{1}{3}(n_0 - n_2) + Hn_4\right]\right\}^\cdot = c_s^2\frac{\Delta}{a^2}(\varphi - H\chi) - 4\pi G\left(e + \frac{2}{3}\frac{\Delta}{a^2}\Pi\right) + c_s^2\left(-\frac{1}{4}n_1 + Hn_2\right) + \frac{4\pi G(\mu+p)}{H}\left[\frac{1}{3}(n_0 - n_2) - aHn_6\right], \tag{309}$$

$$\frac{H^2c_s^2}{4\pi G(\mu+p)a^3}\left\{\frac{4\pi G(\mu+p)a^3}{H^2c_s^2}\left[(\varphi - aHv)^\cdot + \frac{H}{\mu+p}\left(e + \frac{2}{3}\frac{\Delta}{a^2}\Pi\right) - \frac{1}{3}(n_0 - n_2) + aHn_6\right] + \frac{a^3}{H}\left(\frac{1}{4}n_1 - Hn_2\right)\right\}^\cdot = c_s^2\frac{\Delta}{a^2}\left(\varphi - aHv - \frac{2H^2}{\mu+p}\Pi\right) + \frac{Hc_s^2\Delta}{4\pi G(\mu+p)a^2}\left[\frac{1}{3}(n_0 - n_2) - Hn_4\right], \tag{310}$$

$$\frac{1+\Delta/(3a^2\dot H)}{a^3}\left\{\frac{a^3}{1+\Delta/(3a^2\dot H)}\left(\dot\varphi_\delta + \frac{H}{\mu+p}e\right) + \frac{a^3}{3\dot H + \Delta/a^2}\left[\frac{1}{4}\dot n_1 + \frac{1}{4}\left(2H + \frac{\Delta}{3Ha^2}\right)n_1 - Hn_3\right] - \frac{1}{3}a^3n_0\right\}^\cdot = -\frac{1 - c_s^2\Delta/(a^2\dot H)}{3 + \Delta/(a^2\dot H)}\frac{\Delta}{a^2}\varphi_\delta + \frac{1}{\mu+p}\frac{\Delta}{3a^2}\left[e + \frac{2}{3}\left(3\dot H + \frac{\Delta}{a^2}\right)\Pi\right] + \frac{\Delta}{9\dot H a^2}\left[\frac{1}{4a^2}\left(\frac{a^2}{H}n_1\right)^\cdot - n_3 - \left(3\dot H + \frac{\Delta}{a^2}\right)n_4\right], \tag{311}$$

$$\frac{1}{a^3}\left\{a^3\left[\dot{\varphi}_\kappa + \frac{H}{3\dot{H}+\Delta/a^2}\left((1+3c_s^2)\left(\frac{\Delta}{a^2}\varphi_\kappa - \frac{1}{4}n_1\right) - 12\pi G e - n_3\right) - \frac{1}{3}n_0\right]\right\}^{\cdot}$$
$$= \frac{\Delta}{3a^2}\left\{-\varphi_\kappa + \frac{1}{3\dot{H}+\Delta/a^2}\left[(1+3c_s^2)\left(\frac{\Delta}{a^2}\varphi_\kappa - \frac{1}{4}n_1\right) - 12\pi G e - n_3\right] - 8\pi G\Pi - n_4\right\}. \quad (312)$$

Equations (309) and (310) follow by combining Eqs. (305) and (306). Equation (311) follows from Eqs. (95), (96), (98), and (99). Equation (312) follows from Eqs. (95), (96), (98), and (99). The perturbed-order variables in Eqs. (311) and (312) are evaluated in the uniform-density gauge ($\delta \equiv 0$), and the uniform-expansion gauge ($\kappa \equiv 0$), respectively.

C. Large-scale solutions

Now, we assume an ideal fluid, thus set $e = 0 = \Pi$. In the large-scale limit, thus ignoring the $\Delta/(aH)^2$-order higher terms, Eqs. (310)–(312) give

$$\dot{\varphi}_v - \frac{1}{3}(n_0 - n_2) + aHn_6 + \frac{Hc_s^2}{4\pi G(\mu + p)} \times \left(-\frac{1}{4}n_1 + Hn_2\right) \propto \frac{H^2c_s^2}{4\pi G(\mu+p)a^3}, \quad (313)$$

$$\dot{\varphi}_\delta + \frac{1}{3\dot{H}}\left(\frac{1}{4}\dot{n}_1 + \frac{1}{2}Hn_1 - Hn_3\right) - \frac{1}{3}n_0 \propto \frac{1}{a^3}, \quad (314)$$

$$\dot{\varphi}_\kappa - \frac{H}{3\dot{H}}\left(\frac{1+3c_s^2}{4}n_1 + n_3\right) - \frac{1}{3}n_0 \propto \frac{1}{a^3}, \quad (315)$$

where the perturbed-order variables in Eq. (313) are evaluated in the comoving gauge ($v \equiv 0$). We already used the behavior of linear-order solutions in Eqs. (303) and (304) in order to show that the right-hand side of Eqs. (310)–(312) vanish. Using the solutions in Eqs. (303) and (304) we can show that

$$(\varphi_v - \varphi_v^2)^{\cdot} + \mathcal{O}(\Delta C^2, \Delta^2 d^2) \propto \frac{H^2c_s^2}{(\mu+p)a^3}, \quad (316)$$

$$(\varphi_\delta - \varphi_\delta^2)^{\cdot} + \mathcal{O}(\Delta C^2, \Delta^2 d^2) \propto (\varphi_\kappa - \varphi_\kappa^2)^{\cdot} + \mathcal{O}(\Delta C^2, \Delta^2 d^2) \propto \frac{1}{a^3}. \quad (317)$$

Thus, we have general large-scale asymptotic solutions

$$\varphi_v - \varphi_v^2 = C(\mathbf{x}) + \frac{\Delta}{4\pi G}d(\mathbf{x})\int^t \frac{c_s^2H^2}{(\mu+p)a^3}dt, \quad (318)$$

$$\varphi_\delta - \varphi_\delta^2 = \varphi_\kappa - \varphi_\kappa^2 = C(\mathbf{x}) + \frac{\Delta}{3}d(\mathbf{x})\int^t \frac{dt}{a^3}, \quad (319)$$

where $C(\mathbf{x})$ and $d(\mathbf{x})$ are integration constants now including the second-order contributions, i.e., $C = C^{(1)} + C^{(2)}$, etc. Ignoring the transient solutions in an expanding phase we have

$$\varphi_v = \varphi_\delta = \varphi_\kappa = C(\mathbf{x}), \quad (320)$$

even to the second order in perturbations in the large-scale limit.

XII. DISCUSSION

In this work we presented pure general relativistic effects of second-order perturbations in the Friedmann cosmological world model. In our previous works we have shown that to the second-order perturbations, the density and velocity perturbation equations of general relativistic zero-pressure, irrotational, single-component fluid in a flat background *coincide exactly* with the ones known in Newton's theory [19]. We also have shown the effect of gravitational waves to the second order, and pure general relativistic correction terms appearing in the third-order perturbations [19,20]. Here, we presented results of second-order perturbations relaxing all the assumptions made in our previous work in [19]. We derived the general relativistic correction terms arising due to (i) pressure, (ii) multicomponent, (iii) background curvature, and (iv) rotation. We also presented a general proof of large-scale conserved behaviors of a curvature perturbation variable in several gauge conditions, now to the second order.

Effects of pressure can be found in Eqs. (130)–(132). As we emphasized, the effect of pressure is generically relativistic even in the background world model and the linear-order perturbations. Still, our equations show the pure general relativistic effects of pressure (including stresses) appearing in the second-order perturbations. Effects of multicomponent fluids can be found in Eqs. (203)–(208). Although these equations apparently show deviations from the Newtonian situation, in Sec. VII D we showed that if we ignore purely decaying terms in an expanding phase the equations are effectively *the same* as in the Newtonian situation. Effects of background spatial curvature K can be read from Eqs. (226) and (227) or Eqs. (220)–(223). Effects of rotational perturbation can be read from Eqs. (261)–(263). In the small-scale limit we showed that, if we ignore the tensor-type perturbation, the equations including the rotation *coincide* with the Newtonian ones even to the second order. For convenience we summarize the relativistic/Newtonian correspondences and pure general relativistic corrections in the second-order perturbations in Table IV.

To the linear order, we have the relativistic/Newtonian correspondences for the density and velocity perturbations

TABLE IV. The relativistic/Newtonian correspondences and pure general relativistic corrections in the second-order scalar-type perturbations.

Situation	Results
Single component: no pressure, $K = 0$, no rotation, no gravitational waves	Relativistic/Newtonian correspondences of density and velocity perturbations Eqs. (17) and (19), Refs. [18,19]
Multiple component: no pressure, $K = 0$, no rotation, no gravitational waves	Relativistic/Newtonian correspondences of density and velocity perturbations Eqs. (210)–(212) and (216)–(218)
Gravitational waves: single and multiple component, $K = 0$, no-pressure, no-rotation	Pure relativistic corrections Eqs. (17)–(19), Refs. [18,19], Eqs. (210)–(212) and (216)–(218)
Pressure: single component, $K = 0$, no rotation, no gravitational waves	Pure relativistic corrections Eqs. (130)–(132)
$K \neq 0$: single component, no pressure, no rotation, no gravitational waves	Pure relativistic corrections Eqs. (226) and (227)
Rotation: single component, $K = 0$, no pressure, no gravitational waves	Pure relativistic corrections Eqs. (261)–(263)
Rotation, small-scale limit: single component, $K = 0$, no pressure, no gravitational waves	Relativistic/Newtonian correspondences of density and velocity perturbations Eqs. (267)–(269)

in a zero-pressure fluid with general curvature; this is true for both the scalar- and vector-type perturbations, see Eqs. (226) and (227) for the scalar-type perturbations, and Eq. (236) for the vector-type perturbations. For the vector-type perturbations, the relativistic result for the velocity perturbation coincides with the Newtonian one in the presence of the anisotropic stress, see Eq. (236). For the scalar-type perturbations the relativistic/Newtonian correspondence for the density and velocity perturbations is valid for multicomponent zero-pressure fluids in the flat background, see Eqs. (210)–(212) and (216)–(218); this is true even to the second order. We emphasize, however, that the relativistic/Newtonian correspondences do not apply to the gravitational potential or space-time metric perturbation even to the linear order, see below Eq. (227) and below Eq. (237). One exception is the case of scalar-type perturbations of a zero-pressure fluid to the linear order in the flat background: in this case, we can identify the relativistic gauge-invariant variable φ_χ which can be identified as the Newtonian gravitational potential divided by c^2 with a negative sign $-\delta\Phi/c^2$, see Eqs. (178), (179), and (182).

Our results may have important practical implications in cosmology and the large-scale structure formation. Our new result showing relativistic/Newtonian correspondence in the zero-pressure irrotational multicomponent fluids is practically relevant in currently favored cosmology where baryon and (cold) dark matter are two important ingredients of the current matter content in addition to the cosmological constant. The equations in our work are valid in the presence of the cosmological constant. A related important result is the relativistic/Newtonian correspondence valid in the presence of rotational perturbation inside the horizon. Thus, inside the horizon scale, even in the presence of rotational perturbations we can still rely on the Newtonian equations to handle quasilinear evolution of large-scale structures. Considering the pure decaying nature of the rotational perturbation in an expanding phase, unless we have a special mechanism to generate it, the rotational perturbation may not be important even near the horizon scale. As the spatial curvature in the present cosmological era is known to be small [34], the possible presence of small spatial curvature may not be important in the second-order perturbations. Still, while the Newtonian equation is exactly valid to the linear order even in the presence of the spatial curvature, the presence of spatial curvature provides nontrivial general relativistic correction terms to the second order in perturbations. Our second-order perturbation equations in the presence of pressure may have an interesting role as we approach the early stage of universe where the effect of radiation becomes important. The importance of pressure to the second-order perturbations, of course, depends on whether nonlinear effects are significant in the early evolution stage of the large-scale structure during the radiation era and in the early matter dominated era. Realistic estimations of the diverse pure general relativistic contributions from the pressure, the curvature, and the rotation are left for future investigations. The complete sets of equations valid to the second order presented in this work can be used for such investigations in the future.

Here, for a balanced view, it is important to emphasize the limitations of our relativistic/Newtonian correspondences. We would like to emphasize that our relativistic/Newtonian correspondences in several situations and pure general relativistic corrections in the context of Newtonian equations are mainly about the dynamic equations of density and velocity perturbations *without* using the gravitational potential or space-time metric perturbations. Therefore, our relativistic/Newtonian correspondences do *not* imply the absence of many space-time (i.e., pure general relativistic) effects like frame dragging, and redshift and deflection of photons even in the cases where the dynamical equations coincide in both Newton's and Einstein's theories. Particularly, the correspondences concerning only the density and velocity perturbations without referring to the accompanying metric perturbations in Einstein's gravity and the gravitational potential in Newton's gravity imply the possible presence of the pure general relativistic effects caused by the curved nature of the space-time metric in Einstein's theory. One simple example is the deflection of light in zero-pressure medium. The relativistic deflection angle differs from the quasi-Newtonian result by a factor two which is exactly caused by the post-Newtonian (i.e., pure general relativistic) effect on the light propagation.

The effect of third-order perturbations of zero-pressure irrotational multicomponent fluids in a flat background is one obvious remaining issue in our series of investigation of nonlinear cosmological perturbations where nontrivial general relativistic effects are expected. In the case of a single fluid we presented the pure general relativistic effects appearing in the third order in [20]. Corresponding results in the case of the multicomponent will be presented in an accompanying paper [35].

We emphasize that our relativistic perturbation approach is valid in fully general relativistic situations, thus including the superhorizon scale, as long as the weakly nonlinear assumption is valid. In the opposite situation where the processes are weakly relativistic but fully nonlinear, the post-Newtonian approximation provides a complementary approach. A systematic formulation of first-order post-Newtonian equations in the Friedmann cosmological background will be presented in an accompanying paper [16].

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