Infrared Divergence of Pure Einstein Gravity Contributions to the Cosmological Density Power Spectrum

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We probe the pure Einstein gravity contributions to the second-order density power spectrum. On the small scale, we discover that Einstein’s gravity contribution is negligibly small. This guarantees that Newton’s gravity is currently sufficient to handle the baryon acoustic oscillation scale. On the large scale, however, we discover that Einstein’s gravity contribution to the second-order power spectrum dominates the linear-order power spectrum. Thus, the pure Einstein gravity contribution appearing in the third-order perturbation leads to an infrared divergence in the power spectrum.

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The large-scale cosmological density power spectrum can be regarded as one of the main pillars of modern cosmology where theories meet with observations. Recent discovery of the baryon acoustic oscillation (BAO) near the $100h^{-1}$ Mpc scale [1] has spurred the renewed interest of the cosmology community on the importance of detailed theoretical studies of the physics behind the BAO including the initial spectrum and the nonlinear process; $h$ is a Hubble constant in the unit of $100$ km/sec/Mpc. The BAO provides an important distance scale of the sound horizon size at the baryon decoupling epoch, which is completely set by the anisotropic angular power spectrum of cosmic microwave background radiation (CMB), and hence, can be used as a standard ruler. This standard ruler encoded in the power spectrum provides the measurement of the angular diameter distance and the Hubble expansion rate by which we can constrain the properties of dark energy. Recent studies show that nonlinear (next-to-leading-order) processes in the power spectrum are important in the theoretical analysis of the BAO phenomenon, especially in the context of planned precise observations of the high redshift galaxy surveys [2].

Up until now, however, all studies of the cosmological power spectra are treated in the context of Newton’s gravity. The linear-order cosmological perturbation was first handled in Einstein’s gravity by Lifshitz in 1946 [3]. Later a Newtonian study was made by Bonnor in 1957 [4], and it was shown that the Newtonian density perturbation equation coincides exactly with the relativistic result in the zero-pressure limit. We have recently shown in [5] that the density perturbation equation in Einstein’s gravity coincides exactly with the previously known result in Newton’s gravity [6] even to the second-order: we call it a relativistic-Newtonian correspondence to the second-order perturbation. This is a nontrivial result which is available in the temporal comoving gauge with suitable identifications of relativistic metric and matter variables. In the zero-pressure case the Newtonian hydrodynamics is known to be closed to the second order in nonlinearity [6]. Thus, any nonvanishing third-order terms in Einstein’s gravity can be naturally identified as pure general relativistic contributions. Again, with the same comoving gauge and a suitable choice of variables, recently we have derived the density perturbation equation with the third-order corrections [7]; all variables in our gauge condition are naturally gauge invariant.

The pure Einstein’s gravity third-order terms contribute to the next-to-leading-order (second-order) density power spectrum. In this Letter, we present the contribution of the pure Einstein’s gravity corrections to density power spectrum. Our results reveal cosmologically important discoveries in both the small scale and the large scale. Particularly surprising is the unexpected surfacing of an infrared divergence due to the general relativistic nonlinear effect in the large scale, which potentially demands a new revised understanding of the current standard paradigm of the large-scale structure formation mainly based on the linear perturbation theory.

We expand the density fluctuation as $\delta \equiv \delta_1 + \delta_2 + \delta_3 + \cdots$, where $\delta(x, t) = \delta g(x, t) m(t)$ is the relative density fluctuation. The density power spectrum becomes

$$P \equiv \langle |\delta(k, t)|^2 \rangle = \langle |\delta_1|^2 \rangle + \langle |\delta_2|^2 \rangle + 2\langle \Re(\delta_1^* \delta_3) \rangle + \cdots
\equiv P_{11} + P_{22} + P_{13} + \cdots,$$

(1)

where $k$ is a wave number in Fourier space and $\langle \rangle$ indicates phase space averaging; $\langle \Re(\delta_1^* \delta_3) \rangle$ vanishes assuming the random phase [8]; a superscript $*$ indicates a complex conjugation. The density power spectrum is introduced as $\langle \delta(k) \delta(k') \rangle = (2\pi)^3 \delta_D(k + k') P(k)$ with $k = |k| \neq |k'|$ and $\delta_D$ a Dirac delta-function, [9]. The $P_{11}$ is the linear-order power spectrum, and $P_{22} + P_{13}$ is the second-order...
(next-to-leading-order) power spectrum. We notice that the third-order pure general relativistic correction terms contribute to \( P_{13} \); thus we decompose it as \( P_{13} = P_{13,\text{Newton}} + P_{13,\text{Einstein}} \). The density and velocity power spectra valid to the next-to-leading-order are presented in [10]. An integration of the density power spectrum in Eq. (19) of [10] gives

\[
\langle |\delta(k, t)|^2 \rangle = \langle |\delta_1(k, t)|^2 \rangle + \frac{k^3}{2 \pi^2} \int_0^\infty dr |\delta_1(kr, t)|^2 \int_{-1}^1 dx |\delta_1(k\sqrt{1+r^2-2rx}, t)|^2 \frac{(3r+7x-10rx^2)}{(1+r^2-2rx)^2} \\
+ \frac{1}{252} \frac{k^3}{(2\pi)^2} |\delta_1(k, t)|^2 \int_0^\infty dr |\delta_1(kr, t)|^2 \left[ -42r^4 + 100r^2 - 158 + \frac{12}{r^2} + \frac{3}{r^2}(r^2-1)^3(7r^2+2)\ln\left| \frac{1+r}{1-r} \right| \right] \\
+ \frac{10}{21} \left( \frac{\ell}{\ell_H} \right)^2 \frac{k^3}{(2\pi)^2} |\delta_1(k, t)|^2 \int_0^\infty dr |\delta_1(kr, t)|^2 \left[ -\frac{41}{6} r^2 - 21 - \frac{45}{4} \frac{1}{r^2} + \frac{9}{8} \ln\left( \frac{1+36r^2}{1-36r^2} \right) \right] \\
x \left( 5r^6 - 13r^4 + 9r^2 + 1 - \frac{2}{r^2} \right) + \frac{3}{16} \left( 43r^6 + 46r^4 - \frac{53}{r^2} - \frac{36}{r^2} \right) \ln\left| \frac{1-r}{1+r} \right| \right] \\
= P_{11} + P_{22} + P_{13,\text{Newton}} + P_{13,\text{Einstein}}
\]

where \( r \equiv k'/k \) and \( x \equiv (k \cdot k')/(kk') \); \( \ell / \ell_H \equiv \dot{a}/(kc) \) is the ratio between a scale \( \ell \equiv a/k \) and the horizon scale \( \ell_H \equiv c/H \) with \( H \equiv \dot{a}/a \), and \( a(t) \) the cosmic scale factor. The second-order power spectrum in Eq. (2) is derived in Einstein’s gravity; as we have shown that results in Einstein’s gravity coincide exactly with the Newton’s one to the second-order perturbations, the power spectrum in Eq. (2) without \( P_{13,\text{Einstein}} \) is exactly valid in Newton’s gravity. The \( P_{22} \) and \( P_{13,\text{Newton}} \) were presented in [11] in the Newtonian context. The pure Einstein’s gravity contribution \( P_{13,\text{Einstein}} \) is the new result in this work; in Eq. (2) we assumes flat matter dominated background. This pure Einstein’s gravity contribution to the one-loop corrected power spectrum is hitherto unknown in the literature based on Newton’s gravity [9]. We note that the pure Einstein’s gravity contribution is multiplied by a \( (\ell / \ell_H)^2 \) factor, thus suppressed far inside the horizon. The relativistic-Newtonian correspondence to the second order is valid in the zero-pressure situation without rotation, no gravitational waves, and spatially flat background, but valid in the presence of the cosmological constant; for the general cases; see [12].

In Fig. 1 we present integration of Eq. (2) by using a realistic linear power spectrum from a concordance cosmology of \( \Lambda \)CDM universe. We calculate the linear power spectrum using the CAMB [13] code with the maximum likelihood cosmological parameters given in Table. I of [14] (“WMAP + BAO + SN”). Since Eq. (2) is valid for

![FIG. 1 (color online). Second-order power spectrum and the contribution from each component of Eq. (2) at \( z = 6 \). Note that we take the absolute value for negative terms, and show with dashed lines. The vertical dotted line shows the wave number corresponding to the current comoving horizon \( (k_H) \). We use \( k_{\text{min}} = 10^{-2} k_H \) and \( k_{\text{max}} = 10^5 \) to evaluate the integration in Eq. (2).](021301-2)
the spatially flat, matter dominated universe, we calculate the second (next-to-leading) order power spectrum at higher redshift (\(z = 6\)) when our Universe is well approximated by flat, matter dominated universe. When calculating the integration, we set the integration lower bound as \(r_{\text{min}} = 10^{-2}k_H/k\), where \(k_H = 1/\ell_H\) is the wave number corresponding to the horizon scale, and upper bound as \(r_{\text{max}} = 10^3/k\). Note that, unlike the Newtonian correction terms, \(P_{22}\) and \(P_{13,\text{Newton}}\), \(P_{13,\text{Einstein}}\) is sensitive to \(r_{\text{max}}\), and logarithmically diverges. It is a problem that the amplitude of the Einstein term contains a logarithmically divergent integral; a logarithmic divergence is also encountered in [15] in considering the backreaction of inhomogeneity on the cosmological expansion [16].

In the small scale it was found in [8] that the leading order of \(P_{22}\) cancels exactly with the leading order of \(P_{13,\text{Newton}}\). Our result in Fig. 1 shows that despite such a cancellation of the leading-order Newtonian contributions, the next leading-order Newtonian contribution is still bigger than the pure general relativistic contribution \(P_{13,\text{Einstein}}\). We find that \(P_{13,\text{Einstein}}\) is roughly 1\% of \(P_{22} + P_{13,\text{Newton}}\) at \(k = 0.1\ h/\text{Mpc}\,\text{, and it becomes smaller and smaller as } k\text{ increases. This may be a good news to the cosmology community based on Newton’s gravity because our result guarantees to use Newton’s gravity in handling the weakly nonlinear processes far inside the horizon including the BAO scale. In the large scale, however, Fig. 1 reveals a completely unexpected surprising result: the pure general relativistic contribution to the second-order power spectrum dominates over the linear-order relativistic-Newtonian power spectrum. We emphasize that previous nonlinear perturbation studies based on Newton’s gravity have been mainly concerned with the small-scale effect where indeed the fluctuations grow from linear to nonlinear phases. We will discuss implications of this new discovery in the following.}

Figure 1 apparently shows that \(P_{13,\text{Einstein}}\) is bigger than even the linear-order density power spectrum \(P_{11}\) in the large scale; \(P_{13,\text{Einstein}}\) is negative in general. It is not necessarily a problem that the Einstein term dominates at the smallest wave number \(k\); this also happens for instance when a fluctuation distribution with no (very little) large-scale power at higher order grows a \(k^4\) tail at small \(k\) [16,17]. It is, however, a problem that the dominant contribution is negative, which is a sign that perturbation theory is at least incomplete; the full power spectrum is of course positive definite [16]. At its face value our result implies that as the second-order effect is larger than the linear one the nonlinear effect of Einstein’s gravity leads to breakdown of the perturbation theory in the large scale but far inside the horizon scale. The situation is particularly awkward because in the equation level the third-order pure Einstein’s gravity correction terms are smaller than the second-order relativistic-Newtonian terms by a factor \(\varphi_r \sim \delta \Phi/c^2\), see Eqs. (1)-(3) in [10]; \(\delta \Phi\) is the Newtonian gravitational potential which is generally quite small, \(\delta \Phi/c^2 \sim 10^{-7}-10^{-5}\). Thus, we naturally anticipate the perturbatively derived third-order solutions from the relativistic-Newtonian second-order equations [these are smaller than the second-order terms by a factor \(\delta \sim (\ell_H/c^2 \delta \Phi)\) are bigger than the pure Einstein’s gravity contributions inside the horizon; see Eq. (2). That is, in the large-scale limit (\(k \to 0\) and \(r \to \infty\)) a naive examination of Eq. (2) shows that both \(P_{13,\text{Newton}}\) and \(P_{13,\text{Einstein}}\) have \(k\) dependence proportional to \(k^{-2} \delta_1(k,t) \int_0^\infty dk'\). However, an asymptotic expansion of \(P_{13,\text{Newton}}\) shows that both the leading-order term (\(-42r^4\)) and even the next-to-leading-order term (\(100r^2\)) cancel away, whereas no such cancellation occurs in \(P_{13,\text{Einstein}}\). This explains the diverging \(k\) dependence of \(P_{13,\text{Einstein}}\) in Fig. 1 compared with the behavior of \(P_{13,\text{Newton}}\). The divergence of \(P_{13,\text{Einstein}}\) is not a real problem as the density fluctuation is described by \(\mathcal{P} = (k^3/2\pi^2)\mathcal{P}\); as we have \(P_{13,\text{Einstein}} \propto k^{-1}\), \(P_{13,\text{Einstein}}\) is convergent in that limit [16].

A similar naive examination of Eq. (2) in the small scale leads us to expect that \(P_{13,\text{Einstein}}\) could be comparable to the Newtonian second-order contributions. That is, in the small-scale limit (\(k \to \infty\) and \(r \to 0\)) a naive examination of Eq. (2) shows that \(P_{13,\text{Newton}}\) has a \(k\) dependence proportional to \(k^4|\delta_1(k,t)|^2 \int_0^\infty dk'\), whereas \(P_{13,\text{Einstein}}\) has a \(k\) dependence proportional to \(k^2|\delta_1(k,t)|^2 \int_0^\infty dk'\). However, as the leading-order terms in \(P_{13,\text{Newton}}\) cancel exactly with the one in \(P_{22}\) [8], we have \(P_{22} + P_{13,\text{Newton}}\) has the same \(k\) dependence as the leading-order term (\(-45/(4r^2)\)) in \(P_{13,\text{Einstein}}\). Thus, we naturally anticipate that \(P_{13,\text{Einstein}}\) could be comparable with the relativistic-Newtonian contribution to the second-order power spectrum in the small scale. However, an asymptotic expansion of \(P_{13,\text{Einstein}}\) shows that both the leading-order term (\(-45/(4r^2)\)) and even the next-to-leading-order term (\(-21\)) cancel away, whereas no such cancellation occurs in \(P_{13,\text{Newton}}\). This explains why \(P_{13,\text{Einstein}}\) is far smaller than the Newtonian contributions in the small scale.

As a matter of fact, we can hardly accept the result that the Einstein’s gravity leads to breakdown of the perturbation expansion in the scale where the perturbation amplitude is known to be near linear. Being confronted by this unexpected situation we have gone through all the algebra several times without finding a mistake in the analytic calculations leading to Eq. (2); for convenience in the close examination of the algebra by the readers see the whole calculation in the supplementary material available in [18]; the numerical code is also available [19].

Besides any potential error made in our part, we point out that the infrared divergence could be due to an improper choice of our gauge condition. Our results are based on the temporal comoving gauge, setting \(\mathcal{P}_0 = 0\). The relativistic-Newtonian correspondence to the second order is available...
only under this gauge. We have continued to use the same gauge condition and identification of perturbation variables [7]. Whether we could find a nondivergent power spectrum in other gauge conditions is an interesting possibility to be investigated in future work.

Recently, Losic and Unruh presented an intriguing possibility of divergent behaviors of second-order perturbations in the quantum generation stage during inflation which might have a close relation to our result [20]. They argued that “a certain nonlocal measure of second-order metric and matter perturbations generically dominates in its amplitude compared to that of the linear-order perturbations during slow-roll inflation.” They further argued that “during slow roll, second-order fluctuations grew large for a class of inflationary models” and “nonlinear, and probably nonperturbative, gravitational effects dominate near slow-roll spacetime, and therefore the linear perturbation theory likely fails in those situations.” The linear power spectrum used in Fig. 1 is based on a near Harrison-Zel’dovich spectrum [21] which naturally arises from quantum fluctuations during the slow-roll inflation era [22]; this is often regarded as the major triumph of the inflation scenario in the early universe and helped to make the inflation scenario a firm theoretical feature in the early universe despite its energy scale far being beyond an experimentally reachable range. Our result in this Letter is analogous to and consistent with the argument made by Losic and Unruh. Based on the near Harrison-Zel’dovich spectrum, we find a possible divergence in the large scale due to the pure Einstein’s gravity effect on the second-order power spectrum. One way out of this conundrum is to have a correct initial power spectrum generated in the inflation era including the role of the second or higher order perturbation theory during slow-roll inflation era.

Divergent results found in the nonlinear perturbations in the large scale of the density power spectrum here and in the seed generation stage of the early inflation era found by Losic and Unruh, if confirmed to be correct, seriously challenge the currently accepted standard paradigm of theoretical cosmology concerning the cosmological structure formation theory and physics in the early universe. A systematic investigation of the second-order perturbation theory during the quantum generation stage based on inflation is needed to resolve the issue raised in [20]. Our result also suggests systematic investigations are to be made concerning the role of third-order perturbations on the CMB temperature anisotropy power spectrum in the large scale.

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[16] This comment is suggested by an anonymous referee.