Minimum Distance between a Canal Surface and a Simple Surface *

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Abstract

The computation of the minimum distance between two objects is an important problem in the applications such as haptic rendering, CAD/CAM, NC verification, robotics and computer graphics. This paper presents a method to compute the minimum distance between a canal surface and a simple surface (that is, a plane, a natural quadric, or a torus) by finding roots of a function of a single parameter. We utilize the fact that the normals at the closest points between two surfaces are collinear. Given the spine curve $C(t)$, $t_{min} \leq t \leq t_{max}$, and the radius function $r(t)$ for a canal surface, a point on the spine curve $C(t_*)$ uniquely determines a characteristic circle $K(t_*)$ on the surface. Normals to the canal surface at points on $K(t_*)$ form a cone with a vertex $C(t_*)$ and an axis which is parallel to $C'(t_*)$. Then we construct a function of t which expresses the condition that the perpendicular from $C(t)$ to a given simple surface is embedded in the cone of normals to the canal surface at points on $K(t)$. By solving this equation, we find characteristic circles which contain the points of locally minimum distance from the simple surface. Based on these circles, we can compute the minimum distance between given surfaces.

Key Words: Minimum distance, canal surface, simple surface, collision detection, haptic rendering.

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1 Introduction

Computing the distance between two objects is an important problem in many fields such as haptic rendering, CAD/CAM, NC verification, robotics and computer graphics. Distance information is needed for collision detection and virtual prototyping in haptic rendering, for interference avoidance in CAD/CAM and NC verification, for robot path planning and path modification in robotics, and for collision detection and physical simulation in computer graphics [4, 5, 6, 9]. Most of the research on distance computation has concentrated on polyhedral objects [1, 2, 6, 7, 12, 13], and there are a few techniques applicable to surfaces [6, 10, 14, 15, 17].

To compute the minimum distance for two free-form surfaces, many researchers have used the fact that the normal vectors at the closest points are parallel to the line which contains both points. The minimum distance between two parametric surfaces $F(u,v)$ and $G(s,t)$ are described by the following system of equations [11]:

$$
\begin{aligned}
(F(u,v) - G(s,t)) \cdot \frac{\partial F}{\partial u} &= 0 \\
(F(u,v) - G(s,t)) \cdot \frac{\partial F}{\partial v} &= 0 \\
(F(u,v) - G(s,t)) \cdot \frac{\partial G}{\partial s} &= 0 \\
(F(u,v) - G(s,t)) \cdot \frac{\partial G}{\partial t} &= 0.
\end{aligned}
$$

Lin et al. [8] solved this system by using the resultant method. To find the closest points between two Bézier patches, Thomas et al. [15] proposed a method to subdivide given patches while the bounding cones of the normal vectors of the two patches intersect. Johnson and Cohen [6] used geometric method to compute the minimum distance between two surfaces. They compute the lower and upper bounds of the distance between surface patches by using the bounding volumes of the patches and a lower-upper bound tree.

By restricting the objects to relatively simple but frequently used surfaces, there has been more efficient ways to compute the minimum distance. Ho et al. [4] proposed an algorithm for interference avoidance between a machining tool and an arbitrary workpiece. They represented the tool as an assembly of implicit surfaces, and the workpiece as a cloud of points. They efficiently detect the interference between the tool and the workpiece by using the fact that the insideness and outsideness of a point versus an implicit equation is determined very rapidly.

Tornero et al. [16] considered objects as a swept volume of spheres, and computed the distance between the objects by finding the two spheres, one from each object, which are closest. They defined the volume swept by a moving sphere whose center and radius is a linear function as a spherical-cone, and the volume bounded by four boundary spherical-cones and two top and bottom planes as a spherical-plane. Then, they presented closed-form solutions to compute the distance between two spherical-cones, and between a spherical-cone and a spherical-plane. Hamlin et al. [3] extended this work to compute the distance between polytope models by representing them as the convex hull of a finite set of spheres.

This paper presents an efficient and robust method to compute the minimum distance between a canal surface and a simple surface. Canal surfaces and simple surfaces are frequently used to represent a tool, a workpiece and an environment in CAD/CAM and haptic rendering. A canal surface is an envelope surface generated by a moving sphere with a varying radius. Tori, pipe surfaces and Dupin cyclides are some examples of the canal surface. A simple surface is either a plane, a natural quadric (a sphere, a cylinder, or a cone) or a torus. Our method computes the minimum distance between a canal surface and a simple surface by solving a function of a single parameter based on the collinear normal condition already mentioned.

This paper is organized as follows. In Section 2, we present some mathematical preliminaries for the canal surface. Section 3 presents a general solution to compute the minimum distance between a canal surface and a simple surface. Sections 4 – 8 derive the functions to compute the minimum distance for each kind of simple surfaces, and provide examples of the computation. Section 9 concludes this paper.

2 Preliminaries

A canal surface is defined by the trajectory of the center $C(t)$ and the function determining the radius $r(t) > 0$, of the moving sphere, as expressed in the equations :

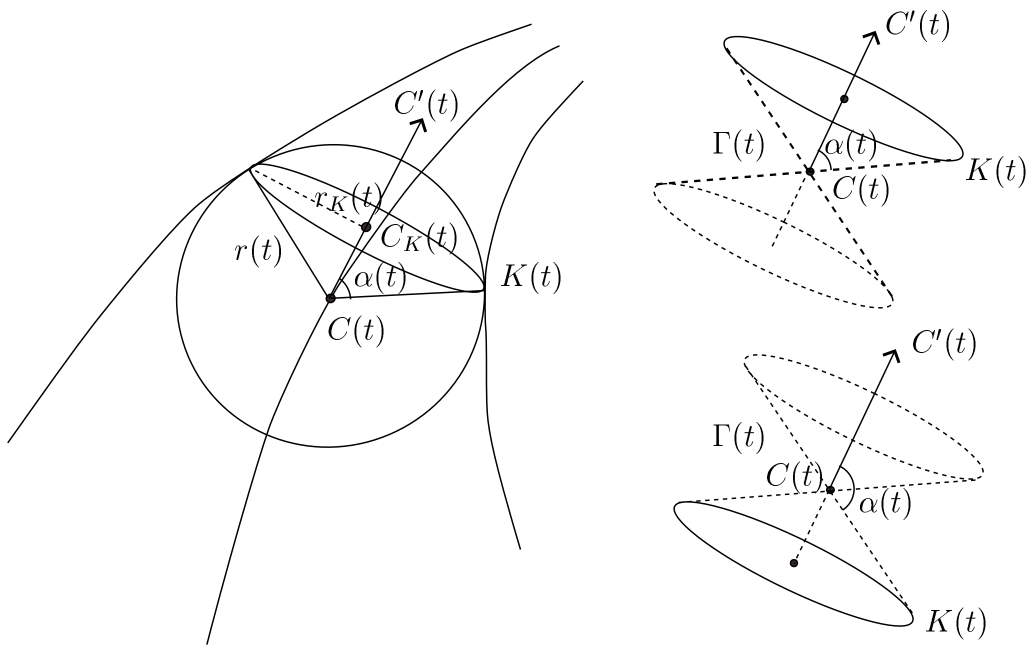
$$\|\mathbf{x} - C(t)\|^2 - r(t)^2 = 0, \tag{1}$$

$$\langle \mathbf{x} - C(t), C'(t) \rangle + r(t)r'(t) = 0, \tag{2}$$

where $\mathbf{x} = (x, y, z)$ is an arbitrary point on the canal surface. Let $\alpha(t)$, $0 < \alpha(t) < \pi$, denote the angle between two vectors $\mathbf{x} - C(t)$ and $C'(t)$. The following relation can then be derived from

Equations (1) and (2) (see Figure 1(a)):

$$\cos \alpha(t) = \frac{\langle \mathbf{x} - C(t), C'(t) \rangle}{\|\mathbf{x} - C(t)\| \|C'(t)\|} = -\frac{r'(t)}{\|C'(t)\|}. \tag{3}$$



(a) (b)

Figure 1: Cone formed by the normals at the points on a characteristic circle.

The moving sphere defined by Equation (1) meets the canal surface at a characteristic circle. We denote by $K(t)$ the characteristic circle which is embedded in the moving sphere by its center from $C(t)$ and radius from $r(t)$. From Equation (3), the center $C_K(t)$, radius $r_K(t)$, and normal $N_K(t)$ to the main plane of the characteristic circle $K(t)$ are derived as follows:

$$\begin{aligned} C_K(t) &= C(t) + r(t)\cos\alpha(t)\frac{C'(t)}{\|C'(t)\|} = C(t) - r(t)r'(t)\frac{C'(t)}{\|C'(t)\|^2} \\ r_K(t) &= r(t)\sin\alpha(t) = r(t)\frac{\sqrt{\|C'(t)\|^2 - r'(t)^2}}{\|C'(t)\|} \\ N_K(t) &= C'(t). \end{aligned}$$

The parametric representation of a canal surface, $S(t, \theta)$, then becomes:

$$S(t, \theta) = C_K(t) + r_K(t)(\cos\theta \mathbf{v}_1(t) + \sin\theta \mathbf{v}_2(t)),$$

where $\mathbf{v}_1(t)$ and $\mathbf{v}_2(t)$ are basis vectors for the plane containing $K(t)$:

$$\begin{aligned}\mathbf{v}_1(t) &= \frac{C'(t) \times C''(t)}{\|C'(t) \times C''(t)\|} \\ \mathbf{v}_2(t) &= \frac{\mathbf{v}_1(t) \times C'(t)}{\|\mathbf{v}_1(t) \times C'(t)\|}.\end{aligned}$$

Figure 1(b) shows now the range of $\alpha(t)$ from Equation (3) varies depending on the location of $C_K(t)$: i) $0 < \alpha(t) \leq \pi/2$, if $C_K(t)$ is on the ray $C(t) + sC'(t)$, where $s \geq 0$, ii) $\pi/2 < \alpha(t) < \pi$, otherwise. We denote the cone which formed by the normals passing through the points on $K(t)$ as $\Gamma(t)$ (see Figures 1(a) and (b)). From the definition of the canal surface, $\Gamma(t)$ is an infinite right circular cone with vertex $C(t)$, and contains $K(t)$. The axis of $\Gamma(t)$ is the line which passes through $C(t)$ and is parallel to $C'(t)$, and the half-angle of the cone is $\mathrm{Min}(\alpha(t),\ \pi - \alpha(t))$.

3 A General Solution

In this section, we compute the minimum distance between an infinite simple surface and a canal surface with a spine curve $C(t) = (x(t), y(t), z(t))$, $t_{min} \leq t \leq t_{max}$, and a radius function $r(t)$. For $t_{min} \leq t \leq t_{max}$, we assume that $C(t)$ is C^2-continuous, $\|C'(t)\|^2 > r'(t)^2$, and $r(t) > 0$. This assumption implies that the canal surface is a regular surface, but the canal surface may have global self-intersections. We may construct a volume which is bounded by the canal surface and two planes which contain $K(t_{min})$ and $K(t_{max})$ and another volume which is bounded by the infinite simple surface. We assume that these two volumes do not intersect each other.

Let us write the perpendicular from a point $C(t)$ to the simple surface as $C(t) + s\mathbf{d}(t)$, where $s \geq 0$ and $\mathbf{d}(t)$ is a direction vector. If $\Gamma(t)$ is the cone which contains the normals to the canal surface at points on $K(t)$, then the necessary condition for $K(t)$ to contain the point of local minimum distance from the simple surface is that the perpendicular $C(t) + s\mathbf{d}(t)$ is embedded in the cone $\Gamma(t)$ (Figure 2). We denote the angle between $\mathbf{d}(t)$ and $C'(t)$ as $\gamma(t)$, and the angle between $\mathbf{x} - C(t)$ and $C'(t)$ as $\alpha(t)$, where $\mathbf{x}$ is an arbitrary point on $K(t)$. Then, we will use the following relation to find the values of t which satisfy the necessary condition :

$$\cos \gamma(t) = \cos \alpha(t),$$

which is reformulated as follows, using Equation (3) :

$$\frac{\langle \mathbf{d}(t), C'(t) \rangle}{\|\mathbf{d}(t)\| \|C'(t)\|} + \frac{r'(t)}{\|C'(t)\|} = 0. \tag{4}$$

When t_* is one of the solutions of Equation (4), the angle between $\mathbf{d}(t_*)$ and $C'(t_*)$ is either $\alpha(t_*)$ or $\pi - \alpha(t_*)$. If the angle between $\mathbf{d}(t_*)$ and $C'(t_*)$ is $\alpha(t_*)$, then the perpendicular $C(t_*) + s\mathbf{d}(t_*)$ intersects with $K(t_*)$ at the point of locally minimum distance from the simple surface (Figure 2(a)). When the angle is $\pi - \alpha(t_*)$, there is no intersection between $C(t_*) + s\mathbf{d}(t_*)$ and $K(t_*)$. In this case, $K(t_*)$ contains a point whose normal is collinear with $\mathbf{d}(t_*)$, but which does not have the point of local minimum distance from the simple surface (Figure 2(b)).

Let T denote the set of roots of the function represented as Equation (4), so that :

$$T = \{ \ t \mid \ \langle \mathbf{d}(t), C'(t) \rangle + r'(t)\|\mathbf{d}(t)\| = 0 \ \}.$$

Then, the minimum distance between the canal surface and the simple surface S is derived as follows:

$$\text{Min}(\text{Dist}(K(t_*), S)), \ \text{for} \ t_* \in \{ \ t \mid t \in T \ \text{and} \ (C(t) + s\mathbf{d}(t)) \cap K(t) \neq \emptyset \ \} \cup \{t_{min}, \ t_{max}\},$$

where $\text{Dist}(K(t_*), S)$ denotes the minimum distance between the circle $K(t_*)$ and S. If $K(t)$ is the circle which contains the point of locally minimum distance from the simple surface, then the distance between $K(t)$ and S is derived as follows:

$$\|C(t) - \mathbf{q}(t)\| - r(t),$$

where $\mathbf{q}(t)$ is the foot-point of the perpendicular from $C(t)$ on to S.

To compute the distance between $K(t_{min})$ (and $K(t_{max})$) and S, we need to compute the distance between an arbitrary circle and S. Although there are some degenerate cases, the computation of the distance between a circle and a simple surface is not a difficult problem. One of the methods to compute the distance between a circle and a simple surface is to use the functions which will be presented in the following sections, with the assumption that the canal surface degenerates to a circle (i.e. canal surface has a circular spine curve and $r(t) = 0$). However, we will skip the details of this computation.

Sections 4 – 8 present the functions corresponding to Equation (4) for each kind of simple surface, and provide examples of computing the minimum distance between a canal surface and a simple surface.

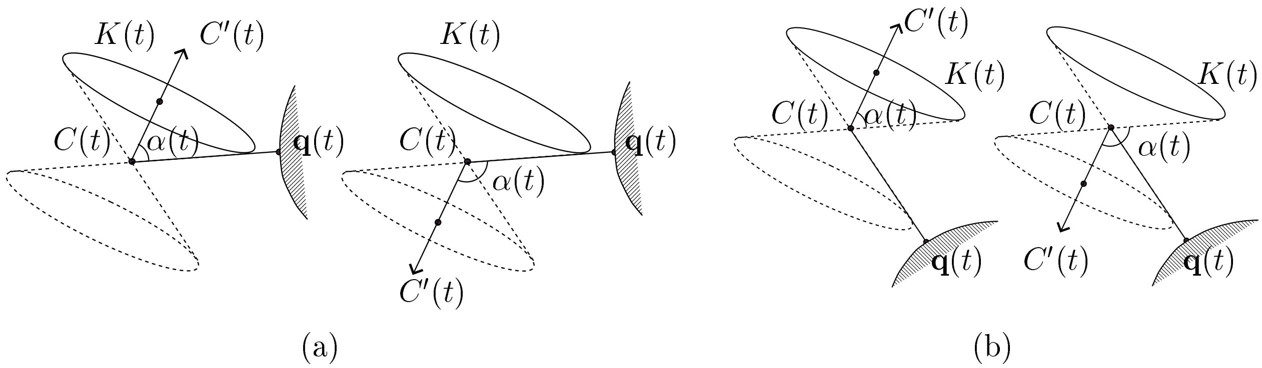


Figure 2: The cases when the angle between $\mathbf{q}(t) - C(t)$ and $C'(t)$ is either $\alpha(t)$ or $\pi - \alpha(t)$.

4 Distance between a Canal Surface and a Plane

Given a plane with reference point $\mathbf{p}$ and unit normal vector $\mathbf{N}$, the foot-point of perpendicular from a point $C(t)$ to the plane, $\mathbf{q}(t)$, is derived as follows:

$$\mathbf{q}(t) = C(t) + \langle \mathbf{p} - C(t), \mathbf{N} \rangle \, \mathbf{N}.$$

We denote the angle between $C'(t)$ and the perpendicular $C(t) + s(\mathbf{q}(t) - C(t))$, $s \geq 0$, as $\gamma(t)$; then the necessary condition for a characteristic circle $K(t)$ to contain the point of locally minimum distance from the plane is that $\gamma(t) = \alpha(t)$. To find the values of t which satisfy this condition, we use the following relation, which represents the condition $\cos\gamma(t) = \cos\alpha(t)$:

$$\frac{\langle \mathbf{q}(t) - C(t), C'(t) \rangle}{\|\mathbf{q}(t) - C(t)\| \|C'(t)\|} \quad = \quad -\frac{r'(t)}{\|C'(t)\|}.$$

When $\|\mathbf{N}\| = 1$, this equation may be simplified to :

$$\langle C'(t), \mathbf{N} \rangle \pm r'(t) = 0. \tag{5}$$

Given $C(t) = (x(t), y(t), z(t))$ and a radius function $r(t)$, where $x(t)$, $y(t)$, $z(t)$ and $r(t)$ are polynomials and the degrees of $C(t)$ and $r(t)$ are m and n respectively, the degree of the function in Equation (5) becomes $\mathrm{Max}(m-1,\ n-1)$, if $r(t)$ is not constant; otherwise it becomes $m-1$.

Figure 3(a) shows the computation of the minimum distance between a plane with reference point $(0,0,0)$ and normal vector $(0,0,1)$ and a canal surface with the spine curve

$$C(t) = (0.2t^3 - 2.5t,\ t^3 + 2t^2,\ 2t^4 - t^3 + 3.5t^2 + 3)$$

and the radius function

$$r(t) = -0.5t + 0.5,$$

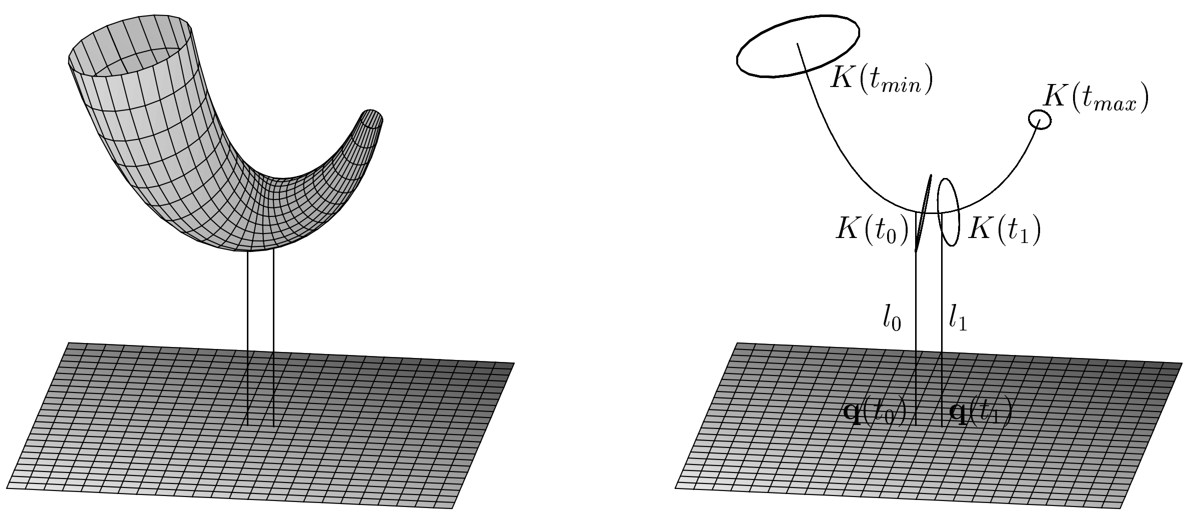
where $-0.7 \le t \le 0.7$. In this example, the solution of Equation (5) is :

$$T = \{-0.06901,\ 0.07328\}.$$

The foot-points of the perpendiculars from $C(t_0)$ and $C(t_1)$, where $t_0 = -0.06901$ and $t_1 = 0.07328$, on to the plane are $\mathbf{q}(t_0)$ and $\mathbf{q}(t_1)$, respectively :

$$\mathbf{q}(t_0) = (0.17246,\ 0.00920,\ 0),$$
$$\mathbf{q}(t_1) = (-0.18312,\ 0.01113,\ 0).$$

In Figure 3(b), there are two line segments l_0 and l_1. The end points of l_0 are $C(t_0)$ and $\mathbf{q}(t_0)$, and those of l_1 are $C(t_1)$ and $\mathbf{q}(t_1)$. The figure shows that the angle between two vectors $\mathbf{q}(t_0) - C(t_0)$ and $C'(t_0)$ is $\alpha(t_0)$, and that between $\mathbf{q}(t_1) - C(t_1)$ and $C'(t_1)$ is $\pi - \alpha(t_1)$. The minimum distance between the given surfaces is the minimum value of the distances between the plane and the three circles $K(t_0)$, $K(-0.7)$, and $K(0.7)$, and is derived as 2.48254.



(a) (b)

Figure 3: Example of computing the minimum distance between a canal surface and a plane.

5 Distance between a Canal Surface and a Sphere

When the center point and radius of a given sphere are $\mathbf{p}$ and R, respectively, the foot-point $\mathbf{q}(t)$ of the perpendicular from a point $C(t)$ on to the sphere is derived as follows:

$$\mathbf{q}(t) = \mathbf{p} + R\frac{C(t) - \mathbf{p}}{\|C(t) - \mathbf{p}\|}.$$

The perpendicular itself can be expressed as $C(t) + s(\mathbf{p} - C(t))$, where $s \geq 0$.

When the angle between $\mathbf{p} - C(t)$ and $C'(t)$ is $\gamma(t)$, the necessary condition for $K(t)$ to contain the point of locally minimum distance from the sphere is $\gamma(t) = \alpha(t)$. To compute the values of t which satisfy the necessary condition, we use the following relation which represents $\cos\gamma(t) = \cos\alpha(t)$:

$$\frac{\langle \mathbf{p} - C(t), C'(t)\rangle}{\|\mathbf{p} - C(t)\|\|C'(t)\|} = -\frac{r'(t)}{\|C'(t)\|}. \tag{6}$$

When $r(t)$ is not constant, Equation (6) can be reformulated by removing the square root :

$$\langle \mathbf{p} - C(t), C'(t)\rangle^2 - r'(t)^2\|\mathbf{p} - C(t)\|^2 = 0. \tag{7}$$

If $r(t)$ is constant, Equation (6) becomes :

$$\langle \mathbf{p} - C(t), C'(t)\rangle = 0. \tag{8}$$

Given $C(t) = (x(t), y(t), z(t))$ and a radius function $r(t)$, where $x(t)$, $y(t)$, $z(t)$ and $r(t)$ are polynomials and the degrees of $C(t)$ and $r(t)$ are m and n respectively, the degrees of the functions in Equation (7) and (8) become $\mathrm{Max}(4m-2,\ 2(m+n-1))$ and $2m-1$ respectively.

Figure 4(a) shows the computation of the minimum distance between a sphere with reference point $(0,0,0)$ and radius 1, and a canal surface with the spine curve

$$C(t) = (0.2t^3 - 2.5t,\ t^3 + 2t^2,\ 2t^4 - t^3 + 3.5t^2 + 3)$$

and the radius function

$$r(t) = -0.5t + 0.5,$$

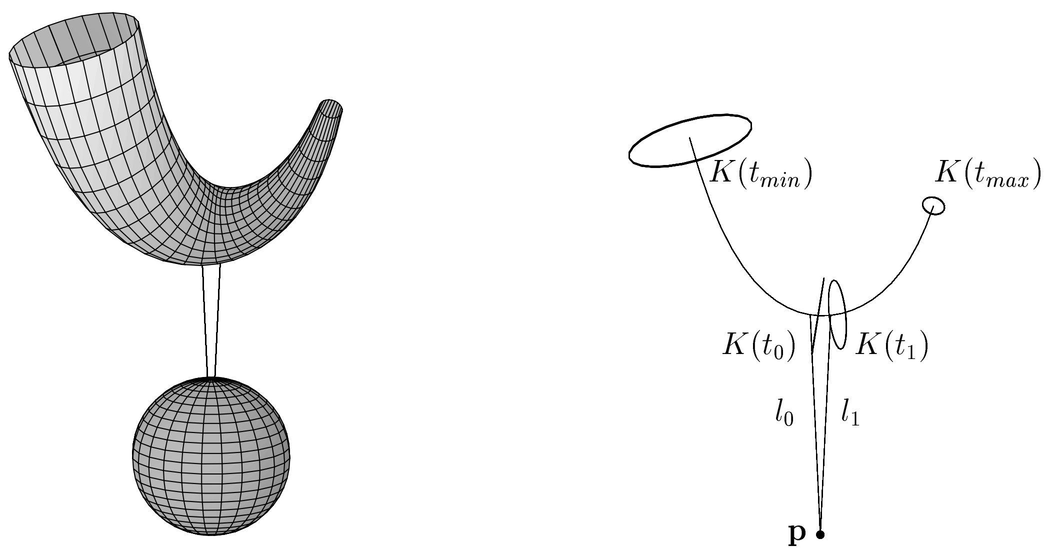
where $-0.7 \leq t \leq 7$. In this example, the solution of Equation (7) is :

$$T = \{-0.05401,\ 0.05599\}.$$

Let us denote -0.05401 and 0.05599 as t_0 and t_1 respectively. In Figure 4(b), the line segment l_0 connects the center point of the given sphere $\mathbf{p}$ and $C(t_0)$. The line segment l_1 connects $\mathbf{p}$ and $C(t_1)$. The foot-points of the perpendiculars from $C(t_0)$ and $C(t_1)$ on to the sphere are $\mathbf{q}(t_0)$ and $\mathbf{q}(t_1)$ respectively:

$$\begin{aligned}\mathbf{q}(t_0) &= (0.04480,\ 0.00188,\ 0.99899)\\ \mathbf{q}(t_1) &= (-0.04643,\ 0.00214,\ 0.99892).\end{aligned}$$

The angle between $C'(t_0)$ and l_0 is $\alpha(t_0)$, and the angle between $C'(t_1)$ and l_1 is $\pi - \alpha(t_1)$. The minimum distance is that between $K(t_0)$ and the sphere, and its value is 1.48641.



(a) (b)

Figure 4: Example of computing the minimum distance between a canal surface and a sphere.

6 Distance between a Canal Surface and a Cylinder

Given a cylinder with axis $\mathbf{p} + v\mathbf{N}$, $v \in I\!R$, and radius R, then the perpendicular from $C(t)$ to the cylinder is :

$$C(t) + s(\bar{\mathbf{q}}(t) - C(t)),$$

where $s \geq 0$ and $\bar{\mathbf{q}}(t) = \mathbf{p} + \langle C(t) - \mathbf{p}, \mathbf{N}\rangle\, \mathbf{N}$. The foot-point $\mathbf{q}(t)$ of the perpendicular from $C(t)$ to the cylinder may be derived as follows :

$$\mathbf{q}(t) = \bar{\mathbf{q}}(t) + R\frac{C(t) - \bar{\mathbf{q}}(t)}{\|C(t) - \bar{\mathbf{q}}(t)\|}.$$

The direction vector of the perpendicular is $\bar{\mathbf{q}}(t) - C(t)$. We denote the angle between $C'(t)$ and $\bar{\mathbf{q}}(t) - C(t)$ as $\gamma(t)$, and then the necessary condition for the characteristic circle $K(t)$ to contain the point of locally minimum distance from given cylinder becomes $\gamma(t) = \alpha(t)$. Incorporating the condition $\cos\gamma(t) = \cos\alpha(t)$, we can then derive the equation :

$$\frac{\langle \bar{\mathbf{q}}(t) - C(t), C'(t)\rangle}{\|\bar{\mathbf{q}}(t) - C(t)\|\|C'(t)\|} = -\frac{r'(t)}{\|C'(t)\|},$$

which leads to the following equation:

$$\langle \bar{\mathbf{q}}(t) - C(t), C'(t)\rangle = -r'(t)\|\bar{\mathbf{q}}(t) - C(t)\|. \tag{9}$$

First, let us consider the case when $r(t)$ is not constant. In this case, replacing $\bar{\mathbf{q}}(t)$ with $\mathbf{p} + \langle C(t) - \mathbf{p}, \mathbf{N}\rangle\, \mathbf{N}$ and removing the square root terms, Equation (9) becomes :

$$\langle \mathbf{p} + \langle C(t) - \mathbf{p}, \mathbf{N}\rangle\, \mathbf{N} - C(t), C'(t)\rangle^2 - r'(t)^2\|\mathbf{p} + \langle C(t) - \mathbf{p}, \mathbf{N}\rangle\, \mathbf{N} - C(t)\|^2 = 0. \tag{10}$$

When $r(t)$ is constant, Equation (9) reduces to :

$$\langle \bar{\mathbf{q}}(t) - C(t), C'(t)\rangle = 0. \tag{11}$$

Given $C(t) = (x(t), y(t), z(t))$ and a radius function $r(t)$, if $x(t)$, $y(t)$, $z(t)$ and $r(t)$ are polynomials and the degrees of $C(t)$ and $r(t)$ are m and n respectively, the degrees of the functions in Equation (10) and (11) become $\mathrm{Max}(4m-2,\ 2(m+n-1))$ and $2m-1$ respectively.

As an example, we compute the minimum distance between a cylinder with reference point $(0,0,0)$, axis parallel to $(1,0,0)$, and radius 1, and a canal surface with spine curve

$$C(t) = (0.2t^3 - 2.5t,\ t^3 + 2t^2,\ 2t^4 - t^3 + 3.5t^2 + 3)$$

and radius function

$$r(t) = -0.5t + 0.5,$$

where $-0.7 \leq t \leq 0.7$. Figure 5(a) shows these canal surface and cylinder.

The solution of Equation (10) is derived as:

$$T = \{-0.06891, 0.07311\}.$$

Let us denote -0.06891 and 0.07311 as t_0 and t_1 respectively. The foot-points of the perpendiculars from $C(t_0)$ and $C(t_1)$ on to the cylinder are $\mathbf{q}(t_0)$ and $\mathbf{q}(t_1)$ respectively, with the following coordinates :

$$\begin{aligned}\mathbf{q}(t_0) &= (0.17220,\ 0.00304,\ 1.00000)\\ \mathbf{q}(t_1) &= (-0.18270,\ 0.00367,\ 0.99999).\end{aligned}$$

Figure 5(b) shows the spine curve of the canal surface and the axis of the cylinder. The line segment l_0 connects the points $\bar{\mathbf{q}}(t_0)$ and $C(t_0)$, and the angle between the vectors $\bar{\mathbf{q}}(t_0) - C(t_0)$ and $C'(t_0)$ is $\alpha(t_0)$. The line segment l_1 connects $\bar{\mathbf{q}}(t_1)$ and $C(t_1)$, and the angle between the vectors $\bar{\mathbf{q}}(t_1) - C(t_1)$ and $C'(t_1)$ is $\pi - \alpha(t_1)$. The minimum distance between given canal surface and cylinder is derived as the expression $\|\mathbf{C}(t_0) - \mathbf{q}(t_0)\| - r(t_0)$, and in this case its value is 1.48255.

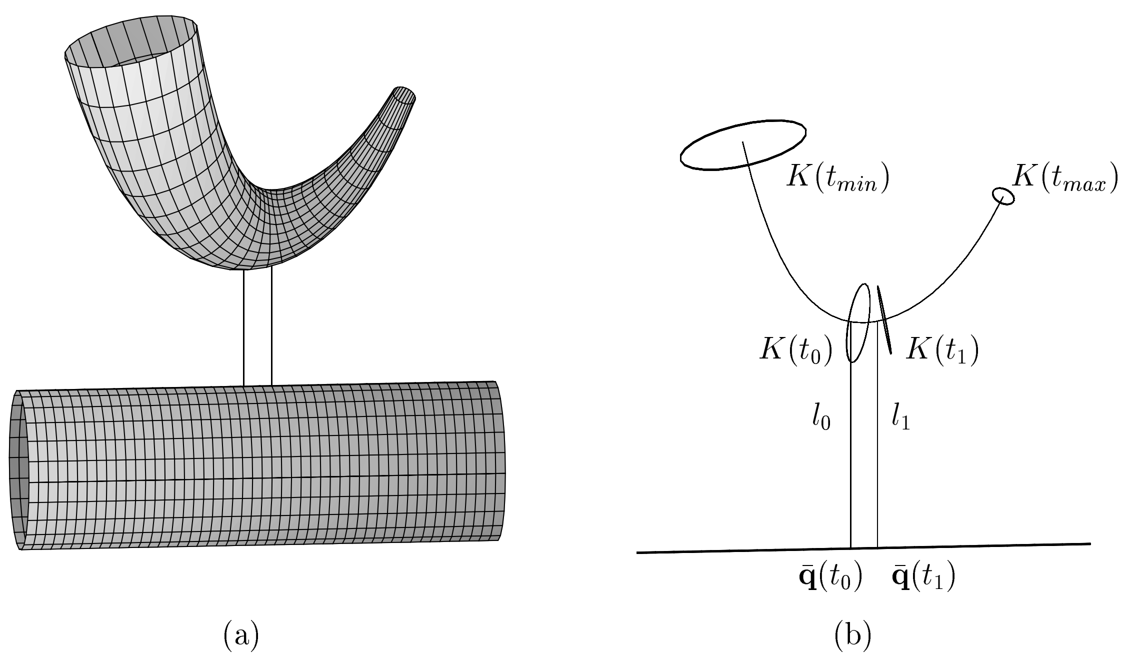


Figure 5: Example of computing the minimum distance between a canal surface and a cylinder.

7 Distance between a Canal Surface and a Cone

Given a canal surface and a cone with vertex $\mathbf{O}$, axis $\mathbf{O} + v\mathbf{N}$, $v \geq 0$, and half-angle β, we may assume that $\mathbf{O} = (0,0,0)$ and $\mathbf{N} = (1,0,0)$, without loss of generality, by applying a translation and rotation to both surfaces if necessary.

When we compute foot-point of the perpendicular from of a point $C(t_*)$ on to given cone, we have to consider the relative position of $C(t_*)$ with respect to the given cone. It is possible that a perpendicular cannot be constructed from $C(t_*)$ on the cone, depending on the position of $C(t_*)$. Let us construct a cone Γ with vertex $\mathbf{O}$, axis $\mathbf{O} - u\mathbf{N}$, $u \geq 0$, and half-angle $\pi/2 - \beta$ (see Figure 6). When $C(t_*)$ is inside Γ, there is no perpendicular to the original cone. Perpendicular and foot-point exist, if $C(t_*)$ is either outside Γ or on the surface of Γ. If $C(t_*)$ is on the surface of Γ, the foot-point of the perpendicular $C(t_*)$ on to the given cone is its vertex $\mathbf{O}$. If $C(t_*)$ is outside Γ, then the foot-point is on the surface of the given cone.

The minimum distance between the given cone and the subpatch of the canal surface, whose spine curve is inside Γ, is the same as the minimum distance between the subpatch and the vertex $\mathbf{O}$ of the given cone. The vertex may be considered as a sphere with zero radius, and so we can compute the minimum distance between the subpatch of the canal surface and the cone by the method presented in Section 5.

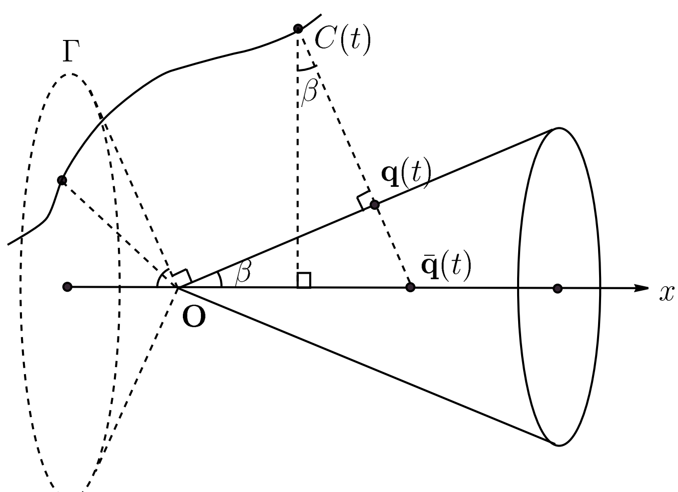


Figure 6: The foot-point of the perpendicular from $C(t)$ on to a cone Γ

Let us assume that the spine curve of the given canal surface $C(t)$ is outside Γ. Then the perpendicular from a point on $C(t)$ to the given cone may be derived as follows (see Figure 6) :

$$C(t) + s(\bar{\mathbf{q}}(t) - C(t)),$$

where $s \geq 0$ and $\bar{\mathbf{q}}(t) = (x(t) + \sqrt{y(t)^2 + z(t)^2}\tan\beta, 0, 0)$. The corresponding foot-point may be computed as follows:

$$\mathbf{q}(t) = \bar{\mathbf{q}}(t) + \frac{C(t) - \bar{\mathbf{q}}(t)}{||C(t) - \bar{\mathbf{q}}(t)||}(x(t) + \sqrt{y(t)^2 + z(t)^2}\tan\beta)\sin\beta.$$

The necessary condition for the characteristic circle $K(t)$ on the canal surface to contain the point of locally minimum distance from the cone is that the angle between $\bar{\mathbf{q}}(t) - C(t)$ and $C'(t)$ should be $\alpha(t)$. Based on this consideration, and incorporating the condition $\cos\gamma(t) = \cos\alpha(t)$, we can derive the equation :

$$\frac{\langle \bar{\mathbf{q}}(t) - C(t), C'(t)\rangle}{||\bar{\mathbf{q}}(t) - C(t)|| ||C'(t)||} = -\frac{r'(t)}{||C'(t)||}. \tag{12}$$

If $r(t)$ is not constant, then Equation (12) can be reformulated as :

$$(\tan\beta\sqrt{y(t)^2 + z(t)^2}x'(t) - y(t)y'(t) - z(t)z'(t))^2 - r'(t)^2(y(t)^2 + z(t)^2)(1 + \tan^2\beta) = 0. \tag{13}$$

If $r(t)$ is constant, then Equation (12) reduces to :

$$\tan\beta\sqrt{y(t)^2 + z(t)^2}x'(t) - y(t)y'(t) - z(t)z'(t) = 0. \tag{14}$$

Given $C(t) = (x(t), y(t), z(t))$ and a radius function $r(t)$, if $x(t)$, $y(t)$, $z(t)$ and $r(t)$ are polynomials and the degrees of $C(t)$ and $r(t)$ are m and n respectively, the degrees of the functions in Equation (13) and (14) become $\mathrm{Max}(8m-4,\ 4(m+n-1),\ 6m+2n-4)$ and $4m-2$ respectively.

Figure 7(a) shows a cone with vertex $(0,0,0)$, axis $(v,0,0)$, $v \geq 0$, and half-angle $\pi/18$, and a canal surface with the spine curve

$$C(t) = (0.2t^3 - 2.5t + 1,\ t^3 + 2t^2,\ 2t^4 - t^3 + 3.5t^2 + 3)$$

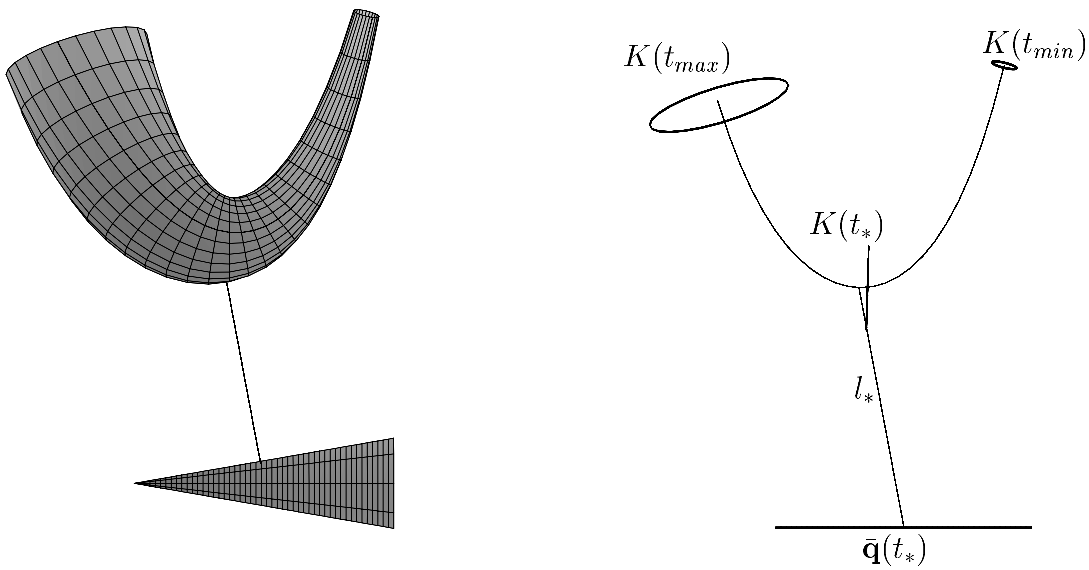
and the radius function

$$r(t) = 0.5t + 0.5,$$

where $-0.7 \leq t \leq 0.7$. The solution of Equation (13) is :

$$T = \{0.00960\}.$$

The foot-point of the perpendicular from $C(t_*)$, $t_* \in T$, on to the cone is $(1.42057,\ 0.00003,\ 0.47911)$. In Figure 7(b), the end points of the line segment l_* are $C(t_*)$ and $\bar{\mathbf{q}}(t_*)$. In this case, the angle between $C'(t_*)$ and $\bar{\mathbf{q}}(t_*) - C(t_*)$ is $\alpha(t_*)$. The minimum distance is formed to be 2.05531.



(a) (b)

Figure 7: Example of computing the minimum distance between a canal surface and a cone.

8 Distance between a Canal Surface and a Torus

Given a canal surface and a torus, we may assume that center of the torus is $\mathbf{O}$ at $(0, 0, 0)$, and that the main plane normal vector $\mathbf{N}$ is at $(0, 0, 1)$ without loss of generality, by applying translation and rotation to both surfaces if necessary. Let R be the major radius of the torus, and δ its minor radius. Note that the perpendicular from an arbitrary point outside the torus, the surface of the torus is also perpendicular to the main circle of the torus, and vice versa.

When a point embedded in the main axis of the torus, $\mathbf{p} = (0, 0, p_z)$, is given, the number of perpendiculars from $\mathbf{p}$ on to the main circle of the torus is infinite, and the main circle is in fact the locus of the foot-points of perpendiculars from $\mathbf{p}$. These points on $C(t)$ which are embedded in the main axis of the torus may be expressed as $C(t_*)$, where $t_* \in \{t \mid x(t) = 0\} \cap \{t \mid y(t) = 0\}$. When $C(t_*) = (0, 0, z(t_*))$, the perpendiculars from $C(t_*)$ to the torus form a right circular cone

Γ with vertex $C(t_*)$, axis $\mathbf{O} + s(\mathbf{O} - C(t_*))$, where $s \geq 0$, and which contains the main circle of the torus. The normals to the points on $K(t_*)$ form another cone $\Gamma(t_*)$ with vertex $C(t_*)$, axis parallel to $C'(t_*)$, and which contains $K(t_*)$ (see Figure 8).

If these two cones $\Gamma(t_*)$ and Γ do not intersect each other except at their vertices, then $K(t_*)$ does not contain the point of locally minimum distance from the torus; that is, when we denote a point on $K(t_*)$ as $\mathbf{p}_K$, and a point on the torus as $\mathbf{p}_T$, there are no pairs $\mathbf{p}_K$ and $\mathbf{p}_T$, whose corresponding normals are collinear. If there exist a $\mathbf{p}_K$ and a $\mathbf{p}_T$ with collinear normals, then the two cones $\Gamma(t_*)$ and Γ intersect at one or two lines. The intersection line is a normal to the canal surface at $\mathbf{p}_K$, and is also normal to the torus at $\mathbf{p}_T$.

Let l denote a line in the intersection between $\Gamma(t_*)$ and Γ. By intersecting l with $K(t_*)$ and the given torus, we derive the points $\mathbf{p}_K$ and $\mathbf{p}_T$ respectively. The normal to the canal surface at $\mathbf{p}_K$ and the normal to the torus at $\mathbf{p}_T$ are collinear. The distance between $K(t_*)$ and the given torus is derived by finding the minimum value of the distances between two points on the given surfaces with collinear normals.

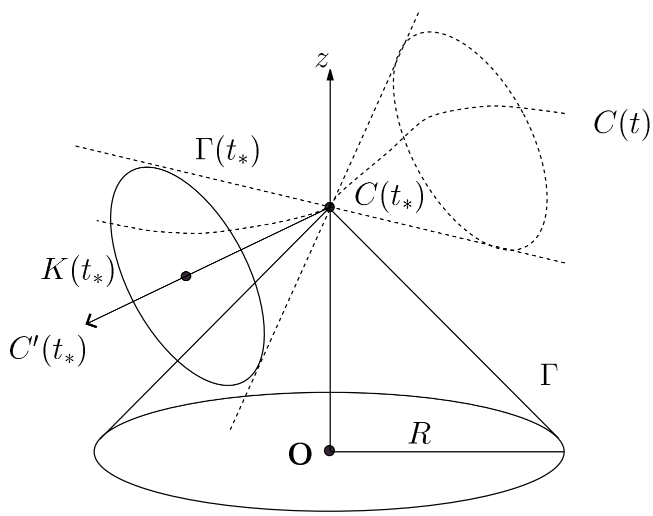


Figure 8: The intersection between the cone consists of perpendiculars from $C(t_*) = (0, 0, z(t_*))$ to the torus and the cone which consists of the normals to the points on $K(t_*)$

Under the assumption that $x(t) \neq 0$ or $y(t) \neq 0$, the perpendicular from a point on $C(t) = (x(t), y(t), z(t))$ to the torus is computed as follows :

$$C(t) + s(\bar{\mathbf{q}}(t) - C(t)),$$

where $s \geq 0$ and $\bar{\mathbf{q}}(t) = \pm R(x(t), y(t), 0)/\sqrt{x(t)^2 + y(t)^2}$. The foot-point of a perpendicular $\mathbf{q}(t)$ from $C(t)$ to the torus may be derived as follows:

$$\mathbf{q}(t) = \bar{\mathbf{q}}(t) + \delta \frac{C(t) - \bar{\mathbf{q}}(t)}{\|C(t) - \bar{\mathbf{q}}(t)\|}.$$

The necessary condition for the characteristic circle $K(t)$ to contain the point of locally minimum distance from the given torus is that the angle between the direction vector of the line $\bar{\mathbf{q}}(t) - C(t)$ and $C'(t)$ is $\alpha(t)$. To find the values of t which satisfy this condition, we expand the equation $\cos\gamma(t) = \cos\alpha(t)$ into the following equation :

$$\frac{\langle \bar{\mathbf{q}}(t) - C(t), C'(t) \rangle}{\|\bar{\mathbf{q}}(t) - C(t)\| \|C'(t)\|} = -\frac{r'(t)}{\|C'(t)\|}. \tag{15}$$

Let us first consider the case when $r(t)$ is not constant. We denote $\bar{\mathbf{q}}(t)$ as $\pm RA(t)/\|A(t)\|$, where $A(t) = (x(t), y(t), 0)$, and reformulate Equation (15) as the function :

$$\begin{aligned} &(R^2 \langle A(t), A'(t) \rangle^2 + \|A(t)\|^2 (\langle C(t), C'(t) \rangle^2 - r'(t)^2 (R^2 + \|C(t)\|^2)))^2 \\ &-4R^2 \|A(t)\|^2 (\langle A(t), A'(t) \rangle \langle C(t), C'(t) \rangle - r'(t)^2 \|A(t)\|^2)^2 = 0. \end{aligned} \tag{16}$$

When $r(t)$ is constant, Equation (15) reduces to :

$$R^2 \langle A(t), A'(t) \rangle^2 - \|A(t)\|^2 \langle C(t), C'(t) \rangle^2 = 0. \tag{17}$$

Given $C(t) = (x(t), y(t), z(t))$ and a radius function $r(t)$, if $x(t)$, $y(t)$, $z(t)$ and $r(t)$ are polynomials and the degrees of $C(t)$ and $r(t)$ are m and n respectively, then the degrees of the functions in Equation (16) and (17) are $\mathrm{Max}(12m - 4,\ 8m + 4n - 4)$ and $6m - 2$ respectively. If $C(t)$ or $r(t)$ is not a polynomial, then the degrees of Equations (16) and (17) are different from the previous derivation. For example, when the given canal surface is a torus (that is, $C(t)$ is a circle and $r(t)$ is constant) the degree of Equation (17) is 8.

Figure 9(a) shows a torus with center point $(0, 0, 0)$, main plane normal vector $(0, 0, 1)$, minor radius 0.3, and major radius 1, and a canal surface with the spine curve

$$C(t) = (0.2t^3 - 2.5t,\ t^3 + 2t^2 - 0.5,\ 2t^4 - t^3 + 3.5t^2 + 2)$$

and radius function

$$r(t) = 0.3t + 0.5,$$

where $-0.7 \leq t \leq 0.7$. In this case, the solution of Equation (16) is

$$T = \{0.02841,\ 0.06299\}.$$

Let us denote these parametric values 0.02841 and 0.06299 as t_0 and t_1 respectively. Figure 9(b) shows the two characteristic circles $K(t_0)$ and $K(t_1)$. The line segment l_0 connects $\bar{\mathbf{q}}(t_0)$ and $C(t_0)$, and the angle between $\bar{\mathbf{q}}(t_0) - C(t_0)$ and $C'(t_0)$ is $\alpha(t_0)$. The line segment l_1 connects $\bar{\mathbf{q}}(t_1)$ and $C(t_1)$, and the angle between $\bar{\mathbf{q}}(t_1) - C(t_1)$ and $C'(t_1)$ is $\alpha(t_1)$. The minimum distance may be derived from $\|\mathbf{q}(t_1) - C(t_1)\| - r(t_1)$, where $\mathbf{q}(t_1) = (-0.28350,\ -0.88567,\ 0.29170)$; its value is 1.25203. Figures 9(c) and (d) show the same surfaces and curves in Figures 9(a) and (b) from a different viewpoint.

9 Conclusions

We have presented a method to compute the minimum distance between a canal surface and a simple surface (that is, a plane, a natural quadric, or a torus) by finding roots of a function of a single parameter. We used the fact that the closest points between two surfaces satisfy the necessary condition that their normals are collinear. When the spine curve $C(t)$ and the radius function $r(t)$ are given for a canal surface, each point $C(t_*)$ on the spine curve uniquely determines a characteristic circle $K(t_*)$ on the surface. The set of normals at points on $K(t_*)$ form a cone with vertex $C(t_*)$ and axis parallel to $C'(t_*)$. From the point $C(t_*)$ we computed a perpendicular to the given simple surface. Then, we constructed a function which satisfies the condition that the perpendicular is embedded in the cone of the normals. By solving this equation, we derived the minimum distance between two surfaces.

All the examples presented in this paper were implemented using tools available in Maple V.

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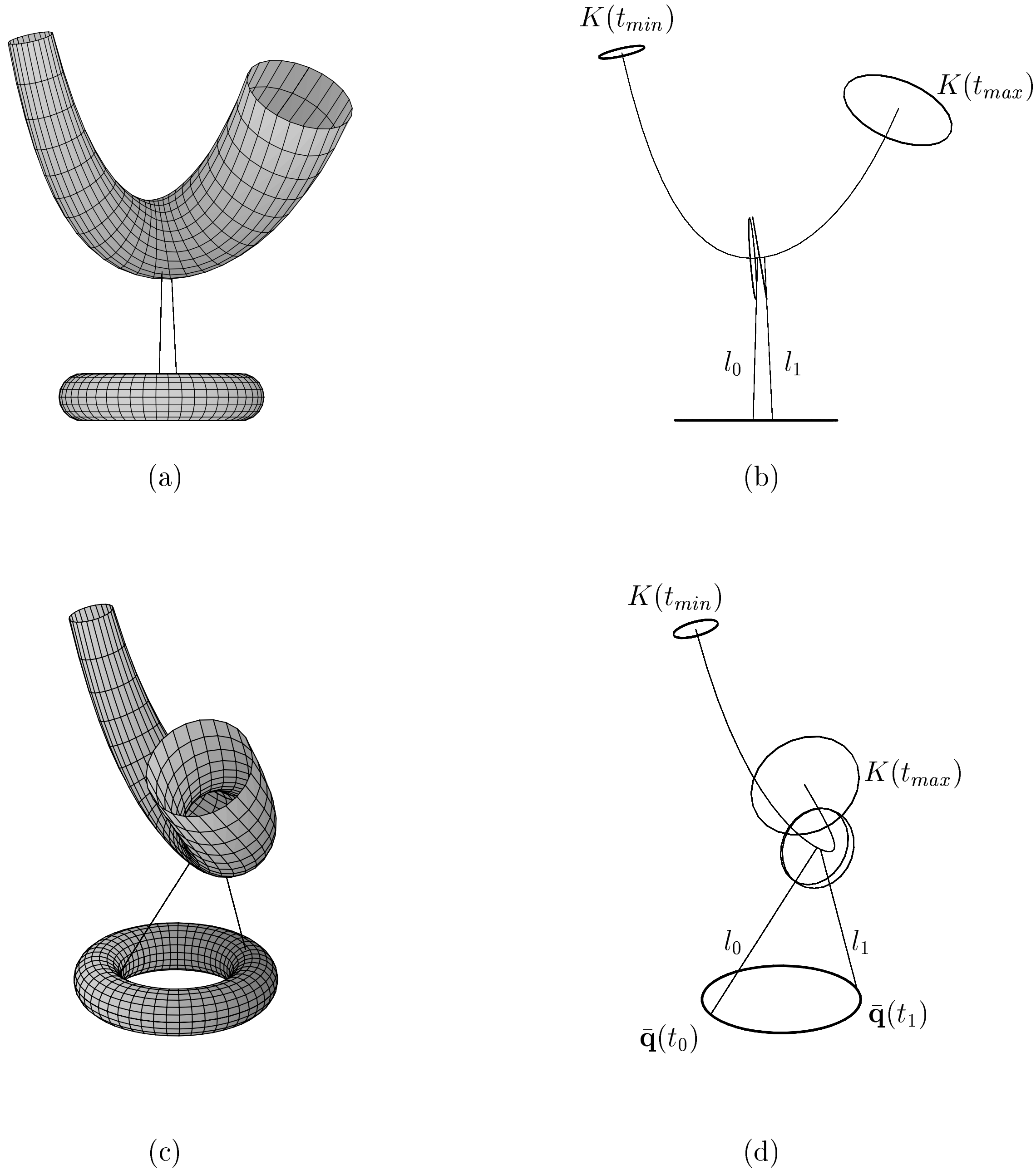

Figure 9: Example of computing the minimum distance between a canal surface and a torus.

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