Computing Isophotes of Surface of Revolution and Canal Surface

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Abstract

Isophote of a surface consists of a loci of surface points whose normal vectors form a constant angle with a given fixed vector. It also serves as a silhouette curve when the constant angle is given as $\pi/2$. We present efficient and robust algorithms to compute isophotes of a surface of revolution and a canal surface. For the two kinds of surfaces, each point on the isophote is derived by a closed-form solution. To find each connected component in the isophote, we utilize the feature of surface normals. Both surfaces are decomposed into a set of circles, where the surface normal vectors at points on each circle construct a cone. The vectors which form a constant angle with given fixed vector construct another cone. We compute the parametric range of the connected component of the isophote by computing the parametric values of the surface which derive the tangential intersection of these two cones.

Keywords: Isophotes, silhouette curve, canal surface, surface of revolution.

1 Introduction

Isophote is one of the characteristic curves on a surface. It consists of points on the surface, at which normal vectors form a constant angle with given fixed vector. The isophote is useful to understand or evaluate the characteristics of the surface. When given surface is C^γ-continuous, the isophote of it is $C^{\gamma-1}$-continuous; thus, the set of isophotes with different constant angles is used to display irregularities of first and second derivatives and Gaussian curvature of a surface [3, 4, 5, 7, 11].

Usually, the isophote of a given surface is computed with two steps: i) computing the normal vector field $\mathbf{N}(u,v)$ of the surface $S(u,v)$, and ii) tracing the surface points whose normal vector $\mathbf{N}(u,v)$ forms a constant angle β, $(0 \le \beta \le \pi/2)$, with given fixed vector $\mathbf{d}$; that is, we have to trace the points on $S(u,v)$ that satisfy the following equation:

$$\frac{\langle \mathbf{N}(u,v), \mathbf{d} \rangle}{||\mathbf{N}(u,v)||} = \cos\beta.$$

Isophote serves as a silhouette curve when the angle β is given as a right angle. In this case, $\mathbf{d}$ is the vector of the line of sight from infinity. The points on the surface $S(u,v)$ which satisfy the following equation construct a silhouette curve :

$$\frac{\langle \mathbf{N}(u,v), \mathbf{d} \rangle}{||\mathbf{N}(u,v)||} = \cos\frac{\pi}{2} = 0.$$

This paper presents efficient and robust algorithms to compute the isophotes of a surface of revolution and a canal surface. Both surfaces are frequently used in CAD/CAM and surface/geometric modeling. Surface of revolution is generated by rotating a planar curve around the rotation axis. Canal surface [10] is an envelope surface of a moving sphere with varying radii. The well–known surfaces such as tori, Dupin cyclides (ring, spindle, and horned) [12] and pipe surfaces [9] are the special cases of the canal surface. Canal surface is often used to blend two surfaces which meet along tangent-discontinuous edges [1, 2, 8]. In this case, isophotes are useful to detect discontinuities of the resulting composite surface at the common boundaries of the patches.

Surface of revolution and canal surface have common properties that the surface can be decomposed into a set of circles, and the surface normals on the same circle construct a cone. We decompose given surface into one–parameter family of circles, $K(t)$. For each circle $K(t_*)$, $t_* \in t$, we derive the equation of the surface normals at the points on it in a simplified form, and present the equation to compute the isophote points on the circle. The isophote points on each circle is derived by a closed-form solution.

The surface normals at the points on a circle $K(t)$ construct a cone $\Gamma(t)$. The vectors which form a constant angle β with given fixed vector $\mathbf{d}$ also construct another cone $\Gamma_{\mathbf{d}}$. Two cones $\Gamma(t)$ and $\Gamma_{\mathbf{d}}$ share the origin as their vertices in $\mathbb{R}^3$ space. The cone $\Gamma_{\mathbf{d}}$ is a fixed one, while the other cone $\Gamma(t)$ varies with respect to the value of t.

When $\Gamma(t)$ and $\Gamma_{\mathbf{d}}$ do not intersect to each other except at the vertices, there is no point on the circle $K(t)$ whose normal vector forms an angle β with the vector $\mathbf{d}$; that is, $K(t)$ does not contain any isophote point on it. When $\Gamma(t)$ and $\Gamma_{\mathbf{d}}$ intersect at two lines, $K(t)$ contains two points whose normal vectors form the angle β with $\mathbf{d}$; that is, there are two isophote points on $K(t)$. Because the axis and half-angle of $\Gamma(t)$ varies continuously, if $K(t_i)$ contains two isophote points and $K(t_j)$ contains no isophote point, a value t_*, where $K(t_*)$ contains only one isophote point, always exists between t_i and t_j. This corresponds to the case when $\Gamma(t_*)$ and $\Gamma_{\mathbf{d}}$ intersect tangentially along a line. The last case is that $\Gamma(t)$ and $\Gamma_{\mathbf{d}}$ share the same axes and half-angles. In this case, the circle $K(t)$ itself is contained in the isophote. Based on these considerations, we find the parametric range of t which derives the connected component of the isophote. The values of t at which $\Gamma(t)$ and $\Gamma_{\mathbf{d}}$ have a tangential intersection classify the range of t at which $K(t)$ contains isophote points or not.

F. Hohenberg [6] and W. Wunderlich [13], which are books on classical descriptive geometry, show the techniques to compute silhouette curves of a surface of revolution and a canal surface. The basic idea to compute isophote which is presented in this paper is based on the techniques in descriptive geometry.

This paper is organized as follows. Section 2 and 3 present efficient and robust algorithms to compute isophotes of a surface of revolution and a canal surface, respectively. Section 4 concludes this paper.

2 Isophote of a Surface of Revolution

Given a surface of revolution and a fixed vector $\mathbf{d}$, we assume that the profile curve $C(t)$ of the surface is a planar curve on xz-plane; that is, $C(t) = (x(t), 0, z(t))$, $t_{min} \leq t \leq t_{max}$, and the surface is generated by rotating $C(t)$ around z-axis. We also assume that given fixed

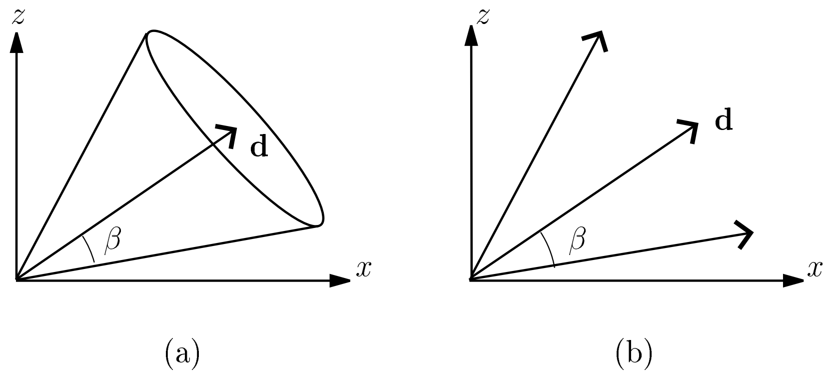


Figure 1: The vectors which form a constant angle β with $\mathbf{d}$

vector $\mathbf{d}$ is contained in xz-plane; that is, $\mathbf{d} = (d_x, 0, d_z)$. These assumptions do not lose the generality, because by applying rotation and translation, an arbitrary surface of revolution and a given vector may be positioned like the above.

The isophote of the surface of revolution consists of a set of points whose normal vectors form a constant angle with the vector $\mathbf{d}$. The vectors which form a constant angle β with $\mathbf{d}$ construct a cone. Then, the vertex of this cone is at the origin, the axis is parallel to $\mathbf{d}$, and the half-angle is β. Figure 1(a) shows this cone.

The surface of revolution generated by rotating the curve $C(t)$ around z-axis is defined by

$$S(t,\theta) = (\cos\theta x(t), \sin\theta x(t), z(t)),$$

where $-\pi \leq \theta < \pi$. The normal vector $\mathbf{N}(t)$ at a point on the curve $C(t)$ is computed as follows:

$$\mathbf{N}(t) = (z'(t), 0, -x'(t)).$$

Then, the normal vector field of the surface $S(t,\theta)$ is computed by rotating the normal vector $\mathbf{N}(t)$:

$$\mathbf{N}(t,\theta) = (\cos\theta z'(t), \sin\theta z'(t), -x'(t)).$$

Figure 2(a) shows a normal vector at a point on the profile curve $C(t)$. Figure 2(b) shows a rotation of the normal vector around z-axis which also represents normal vectors at the points on the cross-section circle of the surface of revolution, where the cross-section circle which contains $C(t_*)$ is represented as: $(\cos\theta x(t_*), \sin\theta x(t_*), z(t_*))$.

The isophote of a surface of revolution $S(t,\theta)$ consists of points which satisfy the following

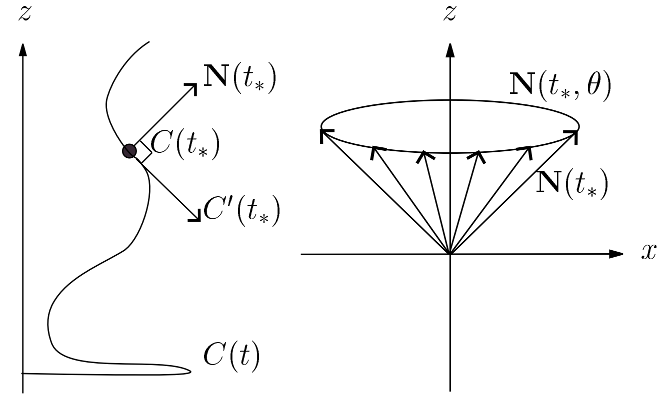


Figure 2: A normal vector at a point on a profile curve and the normal vector field of a cross-section circle of a surface of revolution

condition:

$$\frac{\langle \mathbf{N}(t,\theta), \mathbf{d}\rangle}{\|\mathbf{N}(t,\theta)\|} = \cos\beta,$$

which can be rewritten as follows:

$$\cos\theta = \frac{\cos\beta\sqrt{z'(t)^2 + x'(t)^2} + d_z x'(t)}{d_x z'(t)}. \quad (1)$$

When a fixed value of t, t_*, is given, two isophote points $\mathbf{p}_a(t_*)$ and $\mathbf{p}_b(t_*)$ on the circle $S(t_*, \theta)$ are derived as follows by using Equation (1):

$$\begin{aligned} \mathbf{p}_a(t_*) &= (cx(t_*), \sqrt{1-c^2}x(t_*), z(t_*)) \\ \mathbf{p}_b(t_*) &= (cx(t_*), -\sqrt{1-c^2}x(t_*), z(t_*)), \end{aligned}$$

where

$$c = \frac{\cos\beta\sqrt{z'(t_*)^2 + x'(t_*)^2} + d_z x'(t_*)}{d_x z'(t_*)}.$$

The parametric values of t which classify the real and imaginary roots of θ can be computed by solving the following equation:

$$\frac{\cos\beta\sqrt{z'(t)^2 + x'(t)^2} + d_z x'(t)}{d_x z'(t)} = \pm 1. \quad (2)$$

The degree of Equation (2) is $2(m-1)$, where m is the degree of $C(t)$.

Rather than solving Equation (2) to compute the range of t at which the real component of the isophote exists, we derive more efficient method which uses the geometric properties

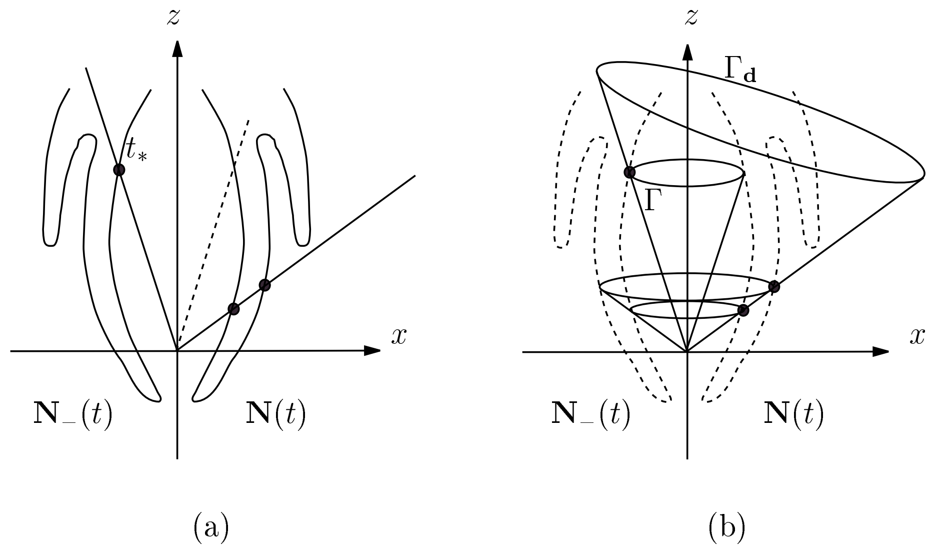


Figure 3: Normal vector of a profile curve and a surface of revolution

of the surface normals as follows. Let $\Gamma_{\mathbf{d}}$ denote the cone which consists of vectors whose angle with the vector $\mathbf{d}$ is β. When we intersect the cone $\Gamma_{\mathbf{d}}$ with xz-plane, following two lines are derived (see Figure 1(b)):

$$(\sin\delta)x - (\cos\delta)z = 0,$$

where

$$\delta = \tan^{-1}(d_z/d_x) \pm \beta.$$

Let us denote the normal vector of the profile curve $C(t)$ as $\mathbf{N}(t)$, where $\mathbf{N}(t) = (z'(t), 0, -x'(t))$. Let $\mathbf{N}_-(t)$ denote the symmetric image of $\mathbf{N}(t)$ about z-axis; that is, $\mathbf{N}_-(t) = (-z'(t), 0, -x'(t))$, and Γ denote a cone which is derived by rotating the normal vector $\mathbf{N}(t_*)$ around z-axis. Notice that Γ is the same as the cone which is derived by rotating $\mathbf{N}_-(t_*)$. If $\mathbf{N}(t) \cup \mathbf{N}_-(t)$ intersects with the lines $(\sin\delta)x - (\cos\delta)z = 0$ at the parametric value of t_*, two cones $\Gamma_{\mathbf{d}}$ and Γ are tangent to each other, and vice versa (see Figure 3). Moreover, this implies that the real component of the isophote starts or ends when $t = t_*$. The intersection between $\mathbf{N}(t)$ and the line $(\sin\delta)x - (\cos\delta)z = 0$ is derived as a following equation:

$$(\sin\delta)z'(t) + (\cos\delta)x'(t) = 0,$$

where $\sin\delta$ and $\cos\delta$ are constant values. Similarly, for $\mathbf{N}_-(t)$, the intersection points are computed by

$$-(\sin\delta)z'(t) + (\cos\delta)x'(t) = 0.$$

Algorithm: Isophote_of_SurfaceOfRevolution

```
Input:
   C(t) = (x(t), 0, z(t)),     /* profile curve of the surface of revolution*/
   d = (d_x, 0, d_z),                                        /* given vector */
   β,                                            /* the angle with vector d */
   Δ                                      /* tolerance for the isophote curve */
begin
/* degenerate cases */
   if d_x = 0 then begin
      for each t_* ∈ { t | x'(t) sqrt(d_z^2 − cos^2 β) ± cos β z'(t) = 0 } do
         if d_z x'(t_*) − cos β sqrt(z'(t_*)^2 + x'(t_*)^2) = 0 then
            draw a circle S(t_*, θ), where 0 ≤ θ < 2π;
      exit;
   end
   for each t_* ∈ { t | z'(t) = 0 } do begin
      if d_z x'(t_*)/||x'(t_*)|| = cos β then
         draw a circle S(t_*, θ), where 0 ≤ θ < 2π;
   end
/* generic case */
   δ = tan^{-1}(d_z/d_x) ± β;
   T = { t | (sin δ)z'(t) ± (cos δ)x'(t) = 0} ∪ {t_min, t_max};
   sort t values in T :  T = { t_i | 0 ≤ i < n };
   for i = 1 to n − 1 do begin
      t_* = (t_{i−1} + t_i)/2;
      if z'(t_*) ≠ 0 and −1 ≤ (cos β sqrt(z'(t_*)^2 + x'(t_*)^2) + d_z x'(t_*)) / (d_x z'(t_*)) ≤ 1 then begin
         New_Curve(C_a); New_Curve(C_b);
         attach two points p_a(t_{i−1}) and p_b(t_{i−1}) to C_a and C_b, respectively;
         Adaptive_Subdivide (t_{i−1}, t_i);
         attach two points p_a(t_i) and p_b(t_i) to C_a and C_b, respectively;
         draw two curves C_a and C_b;
      end
   end
end
```

Algorithm: Adaptive_Subdivide

```
Input:  t_0, t_1;
begin
   t_* = (t_0 + t_1)/2;
   d_a = distance from p_a(t_*) to line segment between p_a(t_0) and p_a(t_1);
   d_b = distance from p_b(t_*) to line segment between p_b(t_0) and p_b(t_1);
   if max(d_a, d_b) ≥ Δ then begin
      Adaptive_Subdivide (t_0, t_*);
      attach two points p_a(t_*) and p_b(t_*) to C_a and C_b, respectively;
      Adaptive_Subdivide (t_*, t_1);
   end
end
```

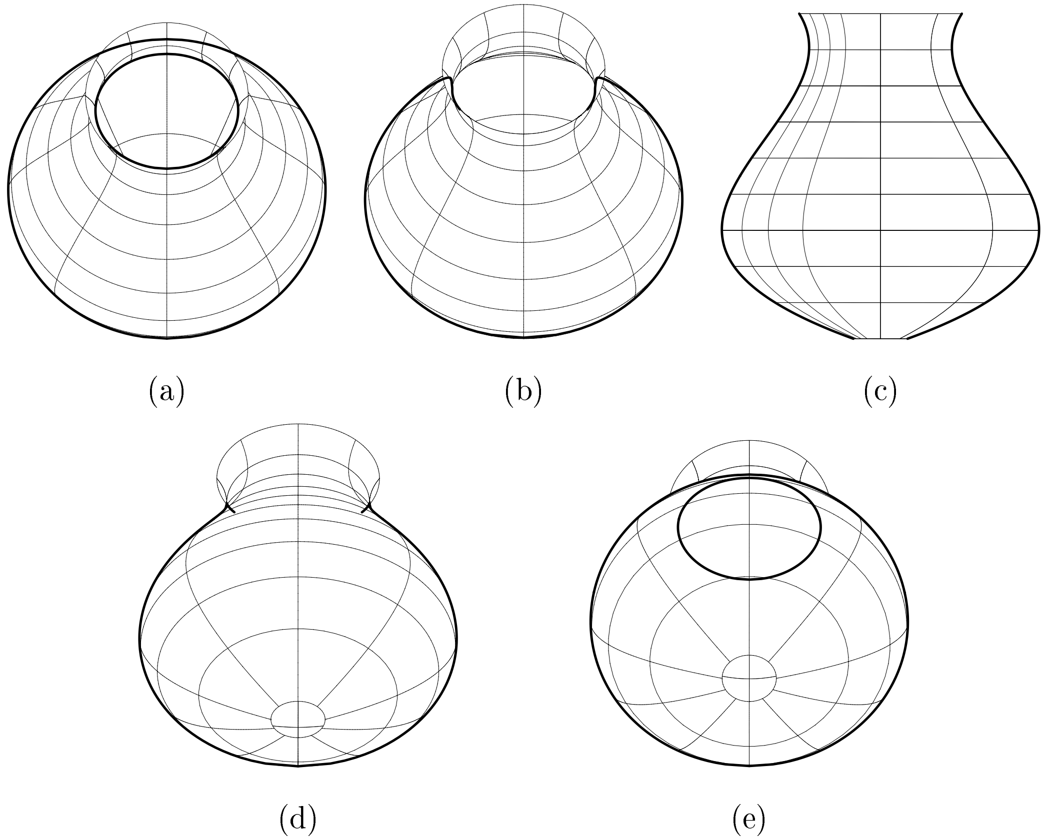
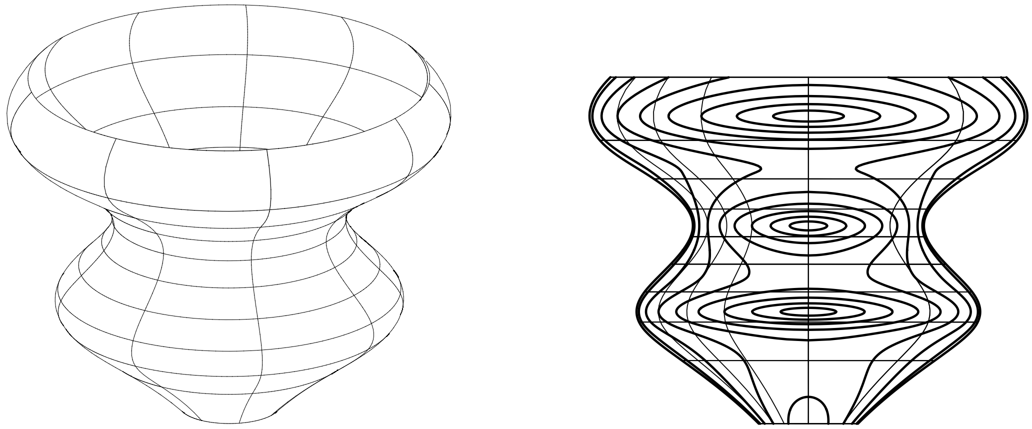


Figure 4: Examples of the silhouette curves (bold curves) of a surface of revolution

The solutions of these equations are the ranges of t at which the real components of the isophote exist. When the degree of $C(t)$ is m, that of this equation is $m-1$; thus, the method using this equation is quite more efficient than the previous method.

Algorithm **Isophote_of_SurfaceOfRevolution** sketches the suggested method with additional considerations on degenerate cases. Figure 4 shows the examples of some silhouette curves of a surface of revolution computed by the suggested method. The profile curve is a cubic Bezier curve, and $\mathbf{d} = (-1,0,-2)$, $(-1,0,-1)$, $(-1,0,0)$, $(-1,0,1)$, and $(-1,0,2)$, respectively, from Figure 4(a) to (e). Notice that the hidden surfaces of given surface of revolution are removed, while the components of the isophote on the hidden surface are not removed.

Figure 5 shows the set of isophotes of a surface of revolution computed with various angles: $\beta = (90-10i)^\circ,\ i = 0..8$.



(a) (b)

Figure 5: The set of isophotes of a surface of revolution: (a) surface of revolution, (b) isophotes (bold curves)

3 Isophote of a Canal Surface

In this section, we present an algorithm to compute the isophote of a canal surface. We derive the parametric representation of a canal surface first. Then, we show the traditional method to compute the isophote. Later, we suggest a more efficient algorithm which uses the geometric property of the canal surface.

3.1 Parametric Representation of a Canal Surface

This section derives a parametric representation of a canal surface. Canal surface is an envelope surface of a moving sphere with varying radii. The center trajectory $C(t)$ and the radius function $r(t)$ of given moving sphere determine a canal surface. For the canal surface to be regular, we assume that the curve $C(t)$ has C^2-continuity, and the conditions $r(t) > 0$ and $\|C'(t)\|^2 > r'(t)^2$ are satisfied for all values of t.

An arbitrary point $\mathbf{x} = (x, y, z)$ on the canal surface is defined by two equations:

$$\|\mathbf{x} - C(t)\|^2 - r(t)^2 = 0, \tag{3}$$

$$\langle \mathbf{x} - C(t), C'(t) \rangle + r(t)r'(t) = 0. \tag{4}$$

Let $\alpha(t)$ denote the angle between two vectors $\mathbf{x} - C(t)$ and $C'(t)$. The following relation is derived by using Equations (3) and (4) (Refer to Figure 6):

$$\cos\alpha(t) = \frac{\langle \mathbf{x} - C(t), C'(t)\rangle}{\|\mathbf{x} - C(t)\| \|C'(t)\|} = -\frac{r'(t)}{\|C'(t)\|}. \tag{5}$$

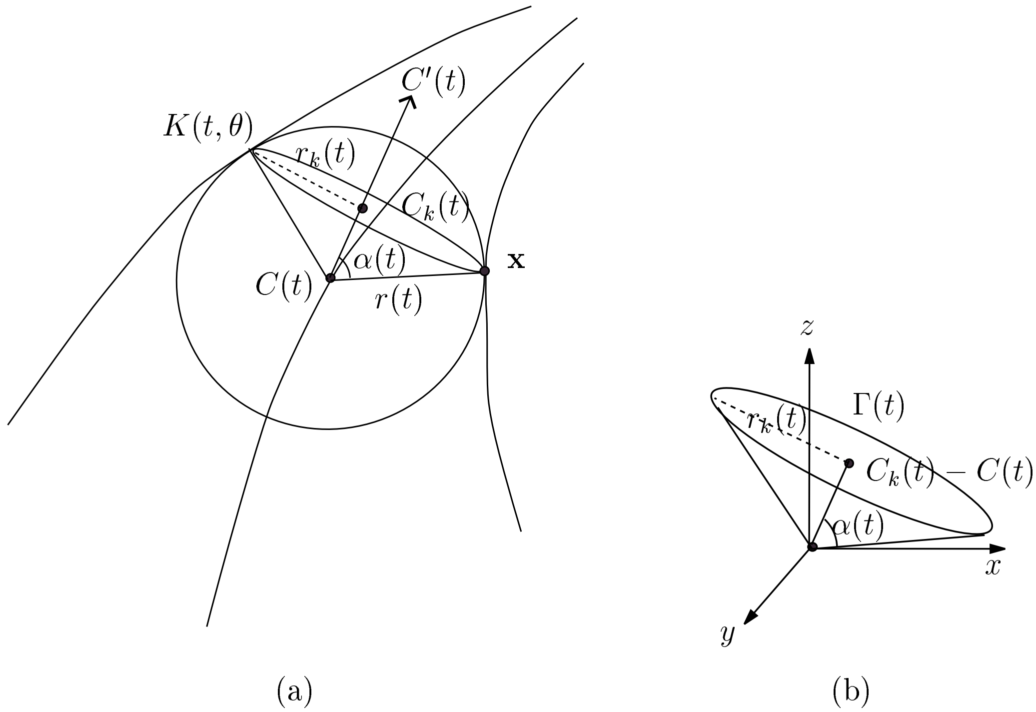


Figure 6: Cone of the normal vectors of a characteristic circle

The moving sphere defined by Equation (3) meets the canal surface at a circle which is called a characteristic circle. When we consider the canal surface as a set of characteristic circles, we can denote the canal surface as $K(t,\theta)$, where $K(t_*,\theta)$ is a characteristic circle contained in a moving sphere with center $C(t_*)$ and radius $r(t_*)$.

By using Equation (5), we derive the center point $C_k(t)$, radius $r_k(t)$, and main plane normal $\mathbf{N}_k(t)$ of a characteristic circle as follows:

$$\begin{aligned} C_k(t) &= C(t) + r(t)\cos\alpha(t)\frac{C'(t)}{\|C'(t)\|} = C(t) - r(t)r'(t)\frac{C'(t)}{\|C'(t)\|^2} \\ r_k(t) &= r(t)\sin\alpha(t) = r(t)\frac{\sqrt{\|C'(t)\|^2 - r'(t)^2}}{\|C'(t)\|} \\ \mathbf{N}_k(t) &= C'(t). \end{aligned}$$

Then, the parametric representation of a canal surface, $K(t,\theta)$, $t_{min} \le t \le t_{max}$ and $0 \le \theta < 2\pi$ is as follows:

$$K(t,\theta) = C_k(t) + r_k(t)(\cos\theta\mathbf{v}_1(t) + \sin\theta\mathbf{v}_2(t)),$$

where

$$\mathbf{v}_1(t) = \frac{C'(t) \times C''(t)}{||C'(t) \times C''(t)||}$$
$$\mathbf{v}_2(t) = \frac{\mathbf{v}_1(t) \times C'(t)}{||\mathbf{v}_1(t) \times C'(t)||}.$$

The vectors $\mathbf{v}_1(t)$ and $\mathbf{v}_2(t)$ are orthogonal unit vectors which determine the orientation of a characteristic circle.

3.2 Traditional Approach

Traditional approach to compute the isophote of a canal surface $K(t, \theta)$ is as follows. Without loss of generality, we may assume that given vector is $\mathbf{d} = (0, 0, 1)$ by applying rotation and translation to both canal surface and the vector $\mathbf{d}$ if necessary. The normal vector field for the surface $K(t, \theta)$ is as follows:

$$\mathbf{N}(t, \theta) = \frac{\partial K(t, \theta)}{\partial t} \times \frac{\partial K(t, \theta)}{\partial \theta}.$$

Then, the isophote with a fixed angle β is derived as follows:

$$\frac{\langle \mathbf{N}(t, \theta), \mathbf{d} \rangle}{||\mathbf{N}(t, \theta)||} = \cos \beta.$$

It is difficult to compute the isophote by solving this equation because the degree of the equation is high. When we consider the canal surface as a set of characteristic circles and use geometric property of the normal vectors at the points on a characteristic circle, there is a more efficient method to compute the isophote of a canal surface. The next section presents the method.

3.3 New Approach

We present an algorithm which computes the isophote of given canal surface $K(t, \theta)$ efficiently. We use the fact that the surface normals at points on a characteristic circle of a canal surface construct a cone. If the normal vector at a point $\mathbf{p}$ forms an angle β with the vector $\mathbf{d}$, the point $\mathbf{p}$ is included in the isophote. Under the assumptions that $0 \leq \beta \leq \pi/2$ and $\mathbf{d} = (0, 0, 1)$, we compute the isophote of a canal surface with the angle β. For the spine curve $C(t) = (x(t), y(t), z(t))$, we derived the center $C_k(t)$, radius $r_k(t)$, and the main plane normal $\mathbf{N}_k(t)$ of the characteristic circle in Section 3.1.

When a point **p** is embedded in the canal surface and a moving sphere with center $C(t)$ and radius $r(t)$ at the same time, we can prove that the canal surface normal at point **p** can be defined as $\mathbf{p} - C(t)$ as follows. According to the definition of the canal surface, the tangent plane of the surface at **p** is also that of the sphere with center $C(t)$ and radius $r(t)$. When we consider the point **p** as a point embedded in the sphere, the vector $\mathbf{p} - C(t)$ is a normal vector of the tangent plane of the sphere at **p**. The point **p** is also on the canal surface; thus, the vector $\mathbf{p} - C(t)$ is also a surface normal vector at point **p** (Refer to Figure 7).

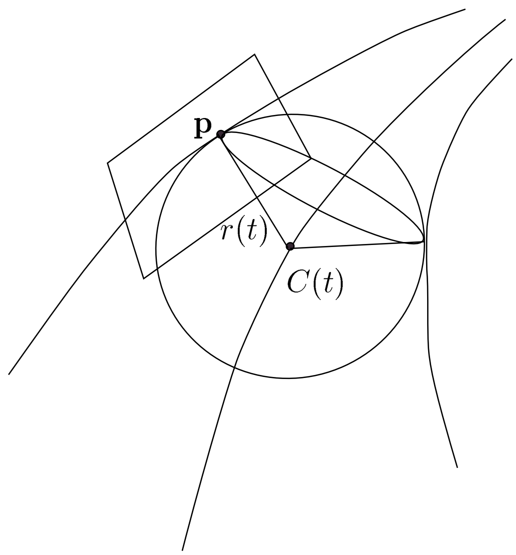


Figure 7: Tangent plane of the canal surface at point **p**

Based on this consideration, the normal vectors $\mathbf{N}(t,\theta)$ at the points on the surface $K(t,\theta)$ are computed as follows :

$$\mathbf{N}(t,\theta) = K(t,\theta) - C(t),$$

where $K(t,\theta) = C_k(t) + r_k(t)(\cos\theta \mathbf{v}_1(t) + \sin\theta \mathbf{v}_2(t))$. The length of the vector $\mathbf{N}(t,\theta)$ is $r(t)$ because $||K(t,\theta) - C(t)|| = r(t)$. Thus, $\mathbf{N}(t,\theta)$ is normalized as follows:

$$\frac{\mathbf{N}(t,\theta)}{||\mathbf{N}(t,\theta)||} = -r'(t)\frac{C'(t)}{||C'(t)||^2} + \frac{\sqrt{||C'(t)||^2 - r'(t)^2}}{||C'(t)||}(\cos\theta \mathbf{v}_1(t) + \sin\theta \mathbf{v}_2(t)).$$

The parameter values of t and θ of isophote satisfy with the following condition:

$$\frac{\langle \mathbf{N}(t,\theta), \mathbf{d}\rangle}{||\mathbf{N}(t,\theta)||} = \cos\beta,$$

which derives the following equation:

$$A(t)\cos\theta + B(t)\sin\theta + D(t) = 0, \tag{6}$$

where

$$\begin{aligned}
A(t) &= \langle \mathbf{v}_1(t), \mathbf{d}\rangle \sqrt{\|C'(t)\|^2 - r'(t)^2} \\
B(t) &= \langle \mathbf{v}_2(t), \mathbf{d}\rangle \sqrt{\|C'(t)\|^2 - r'(t)^2} \\
D(t) &= -r'(t)\frac{\langle C'(t), \mathbf{d}\rangle}{\|C'(t)\|} - \cos\beta\|C'(t)\|.
\end{aligned}$$

We derive following equations from Equation (6):

$$\begin{aligned}
\cos\theta &= \frac{-A(t)D(t) \pm B(t)\sqrt{A(t)^2 + B(t)^2 - D(t)^2}}{A(t)^2 + B(t)^2} \\
\sin\theta &= \frac{-A(t)\cos\theta - D(t)}{B(t)}.
\end{aligned}$$

By using Equation (6), we derive two isophote points $\mathbf{p}_a(t_*)$ and $\mathbf{p}_b(t_*)$ on the characteristic circle $K(t_*, \theta)$ as follows:

$$\begin{aligned}
\mathbf{p}_a(t_*) &= C_k(t_*) + r_k(t_*)(c_a\mathbf{v}_1(t_*) + s_a\mathbf{v}_2(t_*)) \\
\mathbf{p}_b(t_*) &= C_k(t_*) + r_k(t_*)(c_b\mathbf{v}_1(t_*) + s_b\mathbf{v}_2(t_*)),
\end{aligned}$$

where

$$\begin{aligned}
c_a &= \frac{-A(t_*)D(t_*) + \sqrt{A(t_*)^2 + B(t_*)^2 - D(t_*)^2}}{A(t_*)^2 + B(t_*)^2} \\
s_a &= \frac{-A(t_*)c_a - D(t_*)}{B(t_*)},
\end{aligned}$$

and

$$\begin{aligned}
c_b &= \frac{-A(t_*)D(t_*) - \sqrt{A(t_*)^2 + B(t_*)^2 - D(t_*)^2}}{A(t_*)^2 + B(t_*)^2} \\
s_b &= \frac{-A(t_*)c_b - D(t_*)}{B(t_*)}.
\end{aligned}$$

For the vector $\mathbf{d} = (0, 0, 1)$, the set of vectors which form an angle β with $\mathbf{d}$ constructs a cone whose vertex is at the origin, axis is parallel to z-axis, and the half-angle is β. Let's denote this cone as $\Gamma_{\mathbf{d}}$ (Figure 8). When we consider the canal surface as a set of characteristic circles, for a characteristic circle $K(t_*, \theta)$, the set of surface normals at points

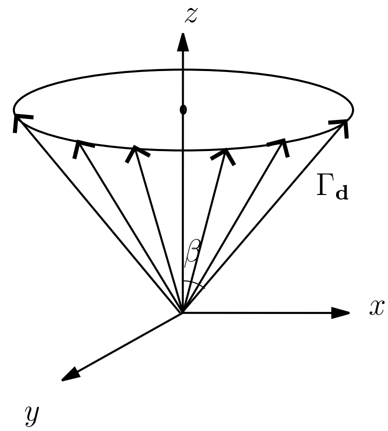


Figure 8: The cone $\Gamma_{\mathbf{d}}$

on it constructs another cone $\Gamma(t_*)$. The vertex of the cone $\Gamma(t_*)$ is at the origin, the axis is $C'(t)$, and the half-angle is $\alpha(t)$ (refer to Section 3.1 and Figures 6(a) and (b)).

If two cones $\Gamma_{\mathbf{d}}$ and $\Gamma(t_*)$ do not intersect each other (Figure 9(a)), the characteristic circle $K(t_*, \theta)$ does not contain a point included in the isophote. When there are two intersection lines between two cones $\Gamma_{\mathbf{d}}$ and $\Gamma(t_*)$ (Figure 9(b)), there are two points of isophote on the circle $K(t_*, \theta)$. When the intersection contains the vector $\mathbf{N}(t_*, \theta_*)$, the point $K(t_*, \theta_*)$ on the canal surface is embedded in the isophote.

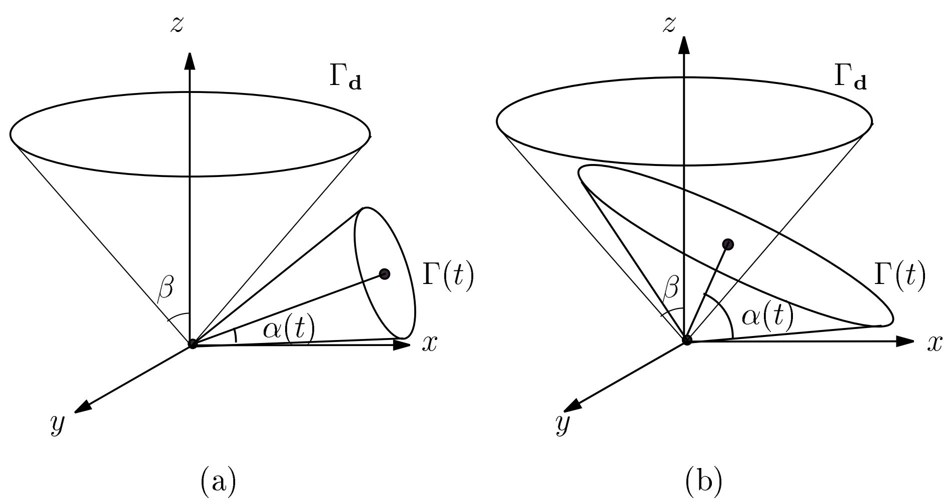


Figure 9: No intersection and intersection cases of $\Gamma(t)$ and $\Gamma_{\mathbf{d}}$

The value of the parameter t which classify the range for Equation (6) to have real roots of θ is computed by finding the value of t at which $\Gamma(t)$ and $\Gamma_{\mathbf{d}}$ intersect tangentially. Two cones have a tangential intersection only when the following condition is satisfied (see Figure

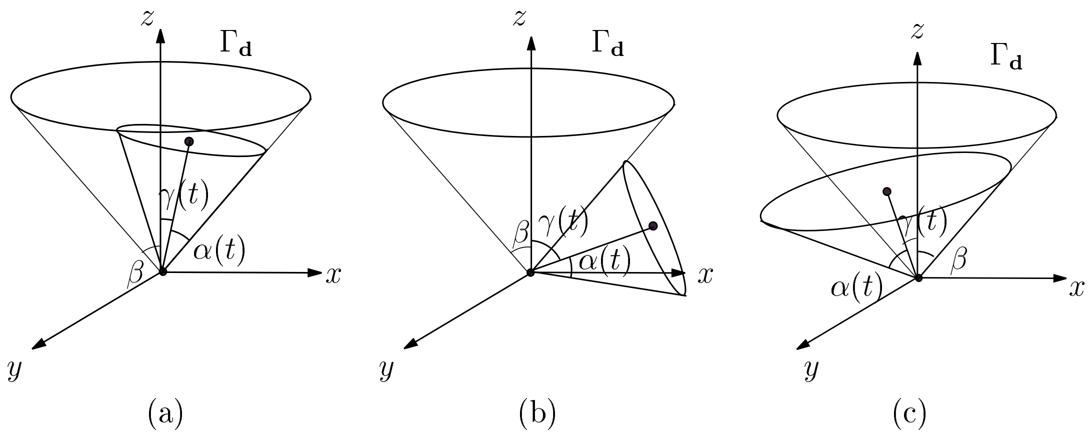


Figure 10: Three cases of $\gamma(t) = |\alpha(t) \pm \beta|$

10):

$$\gamma(t) = |\alpha(t) \pm \beta|,$$

where $\gamma(t)$ is an angle between $C'(t)$ and $\mathbf{d}$, and $0 \leq \gamma(t) < \pi$.

From the fact that $C(t) = (x(t), y(t), z(t))$, $\cos\alpha(t) = \dfrac{-r'(t)}{\|C'(t)\|}$, and $\cos\gamma(t) = \dfrac{\langle C'(t), \mathbf{d}\rangle}{\|C'(t)\|\|\mathbf{d}\|}$, we derive

$$\cos\gamma(t) = \cos(\alpha(t) \pm \beta) = \cos\alpha(t)\cos\beta \pm \sin\alpha(t)\sin\beta,$$

which derives

$$\sin^2\beta(x'(t)^2 + y'(t)^2 + z'(t)^2) - r'(t)^2 - z'(t)^2 - 2z'(t)r'(t)\cos\beta = 0. \qquad (7)$$

By solving Equation (7), we find the parametric range of real components in the isophote efficiently. When the degree of $K(t, \theta)$ in t is m, the degree of Equation (7) is less than m. When the degree of the spine curve $C(t)$ is k and that of the radius function $r(t)$ is n, the degree of Equation (7) is as follows:

$$\max(2(k-1), 2(n-1), k+n-2).$$

Algorithm **Isophote_of_CanalSurface** sketches the method suggested in this section with considerations on the degenerate cases. Figure 11 shows some examples of silhouette curves of the canal surfaces computed by the suggested method. All five canal surfaces are the same canal surface each of which has different orientation, where $\mathbf{d}$ is fixed as $(0, 0, 1)$. Figure 12 illustrates the set of isophotes of a canal surface, where the angle $\beta = 90 - 20i,\ i = 0..4$. The axis curve of the canal surface in Figure 11 and 12 is a cubic B-spline curve with six control points. The degree of the radius curve for both figures is also three.

```
Algorithm: Isophote_of_CanalSurface
Input:  C(t) = (x(t), y(t), z(t)),          /* spine curve of canal surface */
        r(t),                             /* radius function of canal surface */
        d = (0, 0, 1),                               /* given fixed vector */
        β,                                   /* the angle with the vector d */
        Δ                                 /* tolerance for the isophote curve */
begin
  /* degenerate case */
  for each t_* ∈ { t | A(t) = 0 and B(t) = 0 } do
    if D(t_*) = 0 then
      draw a circle K(t_*, θ), 0 ≤ θ < 2π;
  /* generic case */
  T = {t_min, t_max}
    ∪ { t | sin^2 β ||C'(t)||^2 − r'(t)^2 − z'(t)^2 − 2z'(t)r'(t) cos β = 0 };
  sort t values in T :  T = { t_i | 0 ≤ i < n };
  for i = 1 to n − 1 do begin
    t_* = (t_{i−1} + t_i)/2;
    if A(t_*)^2 + B(t_*)^2 − D(t_*)^2 ≥ 0 then begin
      New_Curve(C_a); New_Curve(C_b);
      attach two points p_a(t_{i−1}) and p_b(t_{i−1}) to C_a and C_b, respectively;
      Adaptive_Subdivide (t_{i−1}, t_i);
      attach two points p_a(t_i) and p_b(t_i) to C_a and C_b, respectively;
      draw two curves C_a and C_b;
    end
  end
end

Algorithm: Adaptive_Subdivide
Input:  t_0, t_1;
begin
  t_* = (t_0 + t_1)/2;
  d_a = distance from p_a(t_*) to line segment between p_a(t_0) and p_a(t_1);
  d_b = distance from p_b(t_*) to line segment between p_b(t_0) and p_b(t_1);
  if max(d_a, d_b) ≥ Δ then begin
    Adaptive_Subdivide (t_0, t_*);
    attach two points p_a(t_*) and p_b(t_*) to C_a and C_b, respectively;
    Adaptive_Subdivide (t_*, t_1);
  end
end
```

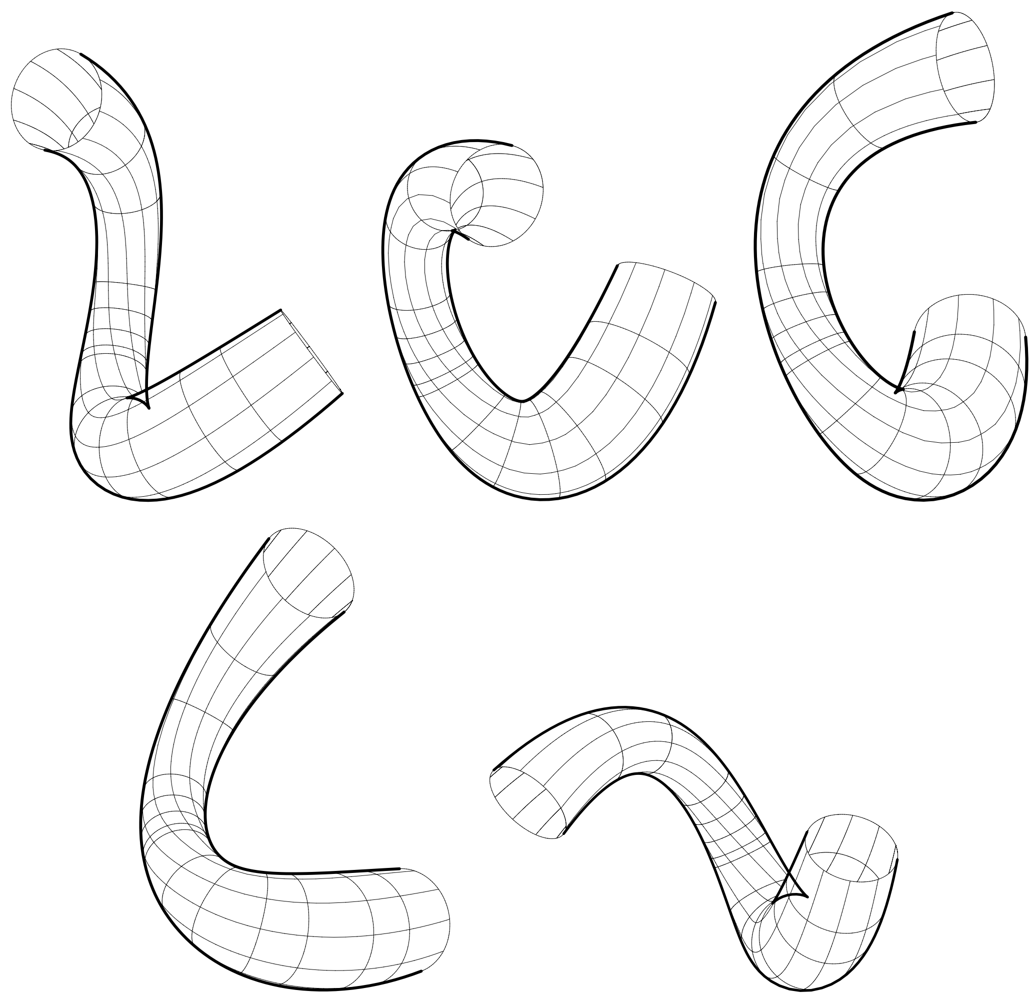


Figure 11: Examples of silhouette curves (bold curves) of canal surfaces

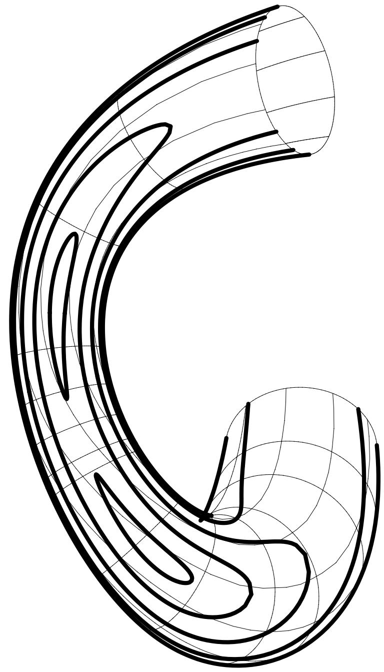


Figure 12: The set of isophotes (bold curves) of a canal surface

4 Conclusion

In this paper, we presented efficient and robust geometric algorithms to compute the isophote of a surface of revolution and a canal surface. These algorithms use the fact that the surfaces are decomposed into a set of circles, and the surface normals on the same circle construct a cone. By considering the relation between this cone and another cone(defined by the constant angle with the given vector $\mathbf{d}$), the range of the isophote is computed, and the isophote itself is traced by using a closed form solution.

5 Acknowledgements

The reviewer's comments about the references [6, 13] are gratefully acknowledged. The authors also thank to the anonymous reviewer whose comments were very useful to improve the presentation of this paper. This research was supported by the Brain Korea 21 Project, and resettlement funds of Ajou University, 2001.

References

[1] R.E. Barnhill, G.E. Farin, and Q. Chen, "Constant radius blending of parametric surfaces," *G.E. Farin, H. Hagen, and H. Noltemeier (Eds) Geometric Modelling* (Springer-Verlag, 1993).

[2] R.A. Farouki and R. Sverrisson, "Approximation of rolling-ball blends for free-form parametric surfaces," *Computer-Aided Design* **28**(11) (1996) 871–878.

[3] N. Guid, Č. Oblonšek, and B. Žalik, "Surface interrogation methods," *Computers and Graphics* **19**(4) (1995) 557–574.

[4] H. Hagen, S. Hahmann, T. Schreiber, Y. Nakajima, B. Wördenweber, and P. Hollemann-Grundstedt, "Surface interrogation algorithms," *IEEE Computer Graphics and Applications* **12(5)** (1992) 53–60.

[5] M. Higashi, T. Kushimoto, and M. Hosaka, "On formulation and display for visualizing features and evaluating quality of free-form surfaces," *Proceedings of EUROGRAPHICS '90* (North-Holland, Amsterdam, 1990) 299–309.

[6] F. Hohenberg, *Konstruktive Geometrie in der Technik*, (Springer, Wien 1956).

[7] M. Hosaka, *Modeling of Curves and Surfaces in CAD/CAM*, (Springer-Verlag, 1992).

[8] G. Lukács, "Differential geometry of G^1 variable radius rolling ball blend surfaces," *Computer Aided Geometric Design* **15** (1998) 585-613.

[9] T. Maekawa, N.M. Patrikalakis, T. Sakkalis, and G. Yu, "Analysis and applications of pipe surfaces," *Computer-Aided Geometric Design* **15** (1998) 437–458.

[10] M. Peternell and H. Pottmann, "Computing rational parametrizations of canal surfaces," *Journal of Symbolic Computation* **23** (1997) 255–266.

[11] T. Poeschl, "Detecting surface irregularities using isophotes," *Computer Aided Geometric Design* **1** (1984) 163–168.

[12] C.-K. Shene, "Blending two cones with Dupin cyclides," *Computer Aided Geometric Design* **15** (1998) 643–673.

[13] W. Wunderlich, *Darstellende Geometrie I*, Bibliographisches Institut Mannheim (1966).